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A 2.5D BEM PROCEDURE BASED ON THE TLM

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Abstract. *The boundary element method (BEM) is currently one of the most widely used numerical methods to solve the wave equation. It is especially useful when the domain of interest is unbounded and the radiation of waves must be accounted for. On the other hand, the thin-layer method (TLM) is an effective tool for the calculation of the Green's function in horizontally layered domains, which makes the TLM method an attractive option for use in the BEM. In this work, the two procedures are linked in the context of structures that are invariant in one direction.*

1 INTRODUCTION

There are situations in which a domain can be idealized as a longitudinally invariant medium, i.e., a structure whose cross section remains constant along a given direction (in this work, the direction y). For instance, in the case of vibrations induced by moving vehicles such as trucks or trains, it is often convenient to idealize the road or the track/tunnel as a structure whose geometry is invariant in the longitudinal direction. In these cases, after performing a Fourier transform of the response fields (from the Cartesian coordinate y to the horizontal wavenumber k_y), the analysis of the three dimensional structure can be reduced to a series of 2D problems. This type of analysis is referred to as a two-and-a-half dimension (2.5D) problem and is normally cast in the wavenumber-frequency domain (k_y, ω).

Additionally, whenever the domain under study is unbounded (e.g., soil-structure interaction problem), the radiation of waves to infinity must be accounted for. The boundary element method (BEM) accounts intrinsically for this condition and therefore is one of the most commonly used tools for these situations. The BEM requires the availability of the so called Green's functions (GF), which are typically those of a whole, uniform space (the Stokes-Kelvin problem), and less often those of layered spaces. In the 2.5D domain, the whole-space GF are known in analytical form [1] while the layered spaces GF can only be obtained via numerical methods. The latter can be obtained either with methods based on transfer matrices [2,3], on stiffness matrices [4] or on the thin-layer method (TLM) [5].

Formulations for the 2.5D BEM based on whole-space GF have previously been presented by Sheng et al [6]. On the other hand, François et al. [7] made use of the GF for layered spaces obtained via the stiffness matrix method. In this work, an alternative formulation of the 2.5D BEM is presented, which is based on GF obtained via the TLM. In comparison with the previous two references, the formulation described in this work has the advantage of not needing any special procedure to deal with the singularities of the GF, since these singularities are dealt with intrinsically in the TLM. Furthermore, the proposed procedure is easier to use since it only relies on the discretization of the layered domain into thin-layers.

This work is organized as follows: in section 2, the TLM is described and it is explained how to obtain the field responses in the wavenumber domain; in section 3, the inverse Fourier transforms are evaluated analytically and it is indicated how to obtain the boundary element coefficients directly; section 4 concludes the work.

2 THIN-LAYER METHOD

The TLM is an efficient semi-analytical method for the calculation of the fundamental solutions of layered media. It consists in expressing the displacement field in terms of a finite element expansion in the direction of layering together with analytical descriptions for the remaining directions. Though initially it was limited to domains of finite depth, paraxial boundaries were developed and coupled to the TLM in order to circumvent this limitation [8]. More recently, perfectly matched layers have been proposed and shown to be more efficient than the paraxial boundaries in the simulation of unbounded domains [9].

In this section, it is explained how to combine the eigenvalues and eigenvectors of the TLM matrices in order to obtain the GFs of a horizontally layered domain. Due to space limitations, only the final expressions are presented. Further information on this method can be found in references [5,10].

2.1 Thin-layer matrices

Following the reference [10], after the layered domain under study is discretized into thin-layers and after application of the principle of weighted residuals, one obtains a system of equations for each thin-layer of the form

$$\bar{\mathbf{P}} = \left[k_x^2 \mathbf{A}_{xx} + k_x k_y \mathbf{A}_{xy} + k_y^2 \mathbf{A}_{yy} + \mathrm{i}\left(k_x \mathbf{B}_x + k_y \mathbf{B}_y \right) + \left(\mathbf{G} - \omega^2 \mathbf{M} \right) \right] \bar{\mathbf{U}} \tag{1}$$

where vector $\bar{\mathbf{P}}$ contains the external tractions $\bar{p}_\alpha\left(k_x, k_y, \omega\right)$ at the nodal interfaces, vector $\bar{\mathbf{U}}$ contains the displacements $\bar{u}_\alpha\left(k_x, k_y, \omega\right)$ at the nodal interfaces ($\alpha = x, y, z$) and the remaining bold variables are matrices that solely depend on the material properties of the thin-layers. These matrices are listed in works [5,10] for the case of cross-anisotropic materials. The variables k_x and k_y represent the horizontal wavenumbers in the transverse and longitudinal directions, respectively, and ω represents the angular frequency.

By means of a similarity transformation, the system of equations (1) can be changed into

$$\tilde{\mathbf{p}} = \left[k_x^2 \mathbf{A}_{xx} + k_x k_y \mathbf{A}_{xy} + k_y^2 \mathbf{A}_{yy} + k_x \tilde{\mathbf{B}}_x + k_y \tilde{\mathbf{B}}_y + \left(\mathbf{G} - \omega^2 \mathbf{M} \right) \right] \tilde{\mathbf{u}} \tag{2}$$

where $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{u}}$ are obtained from $\bar{\mathbf{P}}$ and $\bar{\mathbf{U}}$ by multiplying every third row by $-\mathrm{i}$ and where $\tilde{\mathbf{B}}_x$ and $\tilde{\mathbf{B}}_y$ are obtained from $\mathbf{B}_x$ and $\mathbf{B}_y$ by simply reversing the sign of every third column. Eq. (2) is advantageous over Eq. (1) because the matrices therein are symmetric while in Eq. (1) matrices $\mathbf{B}_x$ and $\mathbf{B}_y$ are not.

After assembling the thin-layer matrices for all the thin-layers, one obtains one global system of equations with the same configuration as Eq. (2), and although it can be easily solved for $\tilde{\mathbf{u}}$, by doing so the TLM will offer no advantage over other methods. Instead, a modal basis is found and with that basis the displacements $\tilde{\mathbf{u}}$ are calculated by modal superposition. The advantage of this procedure is that it allows to calculate analytically the inverse transformation from the domain $\left(k_x, k_y, \omega\right)$ to the domain $\left(x, k_y, \omega\right)$.

2.2 Eigenvalue problem

As described in references [5,10], a modal basis can be found through the solution of the quadratic eigenvalue problem in k of the form

$$\left[k^2 \mathbf{A}_{xx} + k \tilde{\mathbf{B}}_x + \left(\mathbf{G} - \omega^2 \mathbf{M} \right) \right] \boldsymbol{\phi} = \mathbf{0} \tag{3}$$

If the matrices in the eigenvalue problem (3) are rearranged by degrees of freedom (first x, then y and finally z), one observes that the matrices attain the following structures

$$\mathbf{A}_{xx} = \begin{bmatrix} \mathbf{A}_x & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{A}_z \end{bmatrix}, \quad \tilde{\mathbf{B}}_x = \begin{bmatrix} \mathbf{O} & \mathbf{O} & \mathbf{B}_{xz} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{B}_{xz}^T & \mathbf{O} & \mathbf{O} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_x & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{G}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{G}_z \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_x & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{M}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{M}_z \end{bmatrix}$$

Subsequently, the eigenvalue problem (3) can be decoupled into the two eigenvalue problems

$$\left\{k^2\begin{bmatrix}\mathbf{A}_x & \mathbf{O}\\ \mathbf{O} & \mathbf{A}_z\end{bmatrix}+k\begin{bmatrix}\mathbf{O} & \mathbf{B}_{xz}\\ \mathbf{B}_{xz}^T & \mathbf{O}\end{bmatrix}+\begin{bmatrix}\mathbf{C}_x & \mathbf{O}\\ \mathbf{O} & \mathbf{C}_z\end{bmatrix}\right\}\begin{bmatrix}\boldsymbol{\phi}_x\\ \boldsymbol{\phi}_z\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}$$
$$\left(k^2\mathbf{A}_y+\mathbf{C}_y\right)\boldsymbol{\phi}_y=\mathbf{0} \tag{4}$$

which correspond to the generalized Rayleigh and generalized Love eigenvalue problems. The first eigenvalue problem has $2NR$ solutions while the second has $2NL$ solutions. For the calculation of the responses, only the eigenpairs whose eigenvalue has negative imaginary component are considered. This way, only NR solutions of the Rayleigh problem and only NL solutions of the Love problem are needed.

2.3 Modal superposition – response in the $\left(k_x, k_y\right)$ domain

a) Displacements

Based on the eigenpairs, the displacements $\bar{u}_{\alpha\beta}^{mn}$ at the m^{th} nodal interface in direction α due to a unit load applied at the n^{th} nodal interface in direction β are calculated by modal superposition as listed in Table 1, being the coefficients K_{nj} given in Table 2.

$$\bar{u}_{xx}^{mn} = \sum_{j}^{R-\text{modes}} K_{3j}\,\phi_{xj}^m\phi_{xj}^n + \sum_{j}^{L-\text{modes}} K_{4j}\phi_{yj}^m\phi_{yj}^n$$
$$\bar{u}_{yy}^{mn} = \sum_{j}^{R-\text{modes}} K_{4j}\,\phi_{xj}^m\phi_{xj}^n + \sum_{j}^{L-\text{modes}} K_{3j}\,\phi_{yj}^m\phi_{yj}^n$$
$$\bar{u}_{xy}^{mn} = \sum_{j}^{R-\text{modes}} K_{2j}\,\phi_{xj}^m\phi_{xj}^n - \sum_{j}^{L-\text{modes}} K_{2j}\,\phi_{yj}^m\phi_{yj}^n = \bar{u}_{yx}^{mn}$$
$$\bar{u}_{xz}^{mn} = -\mathrm{i}\sum_{j}^{R-\text{mod}} K_{5j}\,\phi_{xj}^m\phi_{zj}^n \qquad \bar{u}_{zx}^{mn} = \mathrm{i}\sum_{j}^{R-\text{modes}} K_{5j}\,\phi_{zj}^m\phi_{xj}^n = -\bar{u}_{xz}^{nm}$$
$$\bar{u}_{yz}^{mn} = -\mathrm{i}\sum_{j}^{R-\text{modes}} K_{6j}\,\phi_{xj}^m\phi_{zj}^n \qquad \bar{u}_{zy}^{mn} = \mathrm{i}\sum_{j=1}^{R-\text{modes}} K_{6j}\,\phi_{zj}^m\phi_{xj}^n = -\bar{u}_{yz}^{nm}$$
$$\bar{u}_{zz}^{mn} = \sum_{j}^{R-\text{modes}} K_{1j}\,\phi_{zj}^m\phi_{zj}^n$$

Table 1: Nodal displacements in frequency-wavenumber domain.

$$K_{1j}=\frac{1}{k^2-k_j^2}, \qquad K_{2j}=\frac{k_xk_y}{k^2\left(k^2-k_j^2\right)}$$
$$K_{3j}=\frac{k_x^2}{k^2\left(k^2-k_j^2\right)}, \qquad K_{4j}=\frac{k_y^2}{k^2\left(k^2-k_j^2\right)}$$
$$K_{5j}=\frac{k_x}{k_j\left(k^2-k_j^2\right)}, \qquad K_{6j}=\frac{k_y}{k_j\left(k^2-k_j^2\right)}$$

Table 2: Kernels K_{nj} $\left(k^2=k_x^2+k_y^2\right)$.

The displacements inside a thin-layer are calculated by vertical interpolation of the nodal values, i.e.,

$$\bar{u}_{\alpha\beta}(z) = \sum_{j=1}^{nn+1} N_j(z)\bar{u}_{\alpha\beta j} \tag{5}$$

being nn the expansion order of the thin-layer, $N_j(z)$ the shape function associated with the j^{th} nodal interface and $\bar{u}_{\alpha\beta j}$ the nodal displacement of the j^{th} nodal interface.

b) Consistent nodal tractions

The consistent nodal tractions acting on one isolated thin-layer can be calculated using Eq. (2) after the nodal displacements of that thin-layer are known. For a load applied in the generic direction β, the vectors $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{u}}$ assume the form

$$\tilde{\mathbf{p}} = \begin{bmatrix} \tilde{\mathbf{p}}_1 \\ \vdots \\ \tilde{\mathbf{p}}_{nn+1} \end{bmatrix} \qquad \tilde{\mathbf{u}} = \begin{bmatrix} \tilde{\mathbf{u}}_1 \\ \vdots \\ \tilde{\mathbf{u}}_{nn+1} \end{bmatrix}$$

being $\tilde{\mathbf{p}}_k = \left\{ \bar{p}_{x\beta k} \quad \bar{p}_{y\beta k} \quad -\mathrm{i}\bar{p}_{z\beta k} \right\}^{\mathrm{T}}$ the modified nodal tractions and $\tilde{\mathbf{u}}_k = \left\{ \bar{u}_{xbk} \quad \bar{u}_{y\beta k} \quad -\mathrm{i}\bar{u}_{z\beta k} \right\}^{\mathrm{T}}$ the modified nodal displacements (the word "modified" is used to refer to the multiplication by factor $-\mathrm{i}$). Furthermore, for the cases in which there is no internal source in the interior of the considered thin-layer, the tractions $\bar{p}_{\alpha\beta k}$ ($k = 2...nn$) are null and only the tractions $\bar{p}_{\alpha\beta 1}$ and $\bar{p}_{\alpha\beta\, nn+1}$ remain non-zero. These non-zero tractions correspond to the tractions that the rest of the domain transmits to the thin-layer through the upper and lower interfaces. By replacing in Eq. (2) the displacements by their modal expansion as given in Table 1, the consistent tractions are obtained also in terms of a modal superposition. Hence, considering a force applied at the global interface n in the direction $\beta = x$, the nodal tractions at a thin-layer are obtained by

$$\begin{aligned}
\tilde{\mathbf{p}} = {} & \mathbf{A}_{xx} \left(\sum_{j=1}^{NR} \phi_{xj}^n \begin{bmatrix} \Gamma_{jR}^{x2} & & \\ & \ddots & \\ & & \Gamma_{jR}^{x2} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{Rj}^l \\ \vdots \\ \mathbf{\Phi}_{Rj}^m \end{bmatrix} + \sum_{j=1}^{NL} \phi_{yj}^n \begin{bmatrix} \Gamma_{jL}^{x2} & & \\ & \ddots & \\ & & \Gamma_{jL}^{x2} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{Lj}^l \\ \vdots \\ \mathbf{\Phi}_{Lj}^m \end{bmatrix} \right) + \\
& \left(k_y \mathbf{A}_{xy} + \tilde{\mathbf{B}}_x \right) \left(\sum_{j=1}^{NR} \phi_{xj}^n \begin{bmatrix} \Gamma_{jR}^{x1} & & \\ & \ddots & \\ & & \Gamma_{jR}^{x1} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{Rj}^l \\ \vdots \\ \mathbf{\Phi}_{Rj}^m \end{bmatrix} + \sum_{j=1}^{NL} \phi_{yj}^n \begin{bmatrix} \Gamma_{jL}^{x1} & & \\ & \ddots & \\ & & \Gamma_{jL}^{x1} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{Lj}^l \\ \vdots \\ \mathbf{\Phi}_{Lj}^m \end{bmatrix} \right) + \\
& \left(k_y^2 \mathbf{A}_{yy} + k_y \tilde{\mathbf{B}}_y + \mathbf{G} - \omega^2 \mathbf{M} \right) \left(\sum_{j=1}^{NR} \phi_{xj}^n \begin{bmatrix} \Gamma_{jR}^{x0} & & \\ & \ddots & \\ & & \Gamma_{jR}^{x0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{Rj}^l \\ \vdots \\ \mathbf{\Phi}_{Rj}^m \end{bmatrix} + \sum_{j=1}^{NL} \phi_{yj}^n \begin{bmatrix} \Gamma_{jL}^{x0} & & \\ & \ddots & \\ & & \Gamma_{jL}^{x0} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{Lj}^l \\ \vdots \\ \mathbf{\Phi}_{Lj}^m \end{bmatrix} \right)
\end{aligned} \tag{6}$$

where

$$\begin{gathered}
\Gamma_{jR}^{xp}\left(k_x, k_y\right) = \begin{bmatrix} k_x^p K_{3j} & 0 & 0 \\ 0 & k_x^p K_{2j} & 0 \\ 0 & 0 & k_x^p K_{5j} \end{bmatrix}, \; \Gamma_{jL}^{xp}\left(k_x, k_y\right) = \begin{bmatrix} k_x^p K_{4j} & 0 & 0 \\ 0 & -k_x^p K_{2j} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathbf{\Phi}_{Rj}^k = \left\{ \phi_{xj}^k \quad \phi_{xj}^k \quad \phi_{zj}^k \right\}^{\mathrm{T}}, \quad \mathbf{\Phi}_{Lj}^k = \left\{ \phi_{yj}^k \quad \phi_{yj}^k \quad 0 \right\}^{\mathrm{T}}, \quad k = l...m
\end{gathered} \tag{7}$$

Similar expressions can be obtained for loads in the y and z directions.

c) Derivatives and internal stresses

In the $\left(k_x,k_y\right)$ domain, the horizontal derivatives of the displacements are obtained by simply multiplying the displacements by $-\mathrm{i}k_x$ or $-\mathrm{i}k_y$, depending on the direction of the derivative. As for the vertical derivatives, they can be obtained by combining the nodal displacements weighted by the derivatives of the associated shape functions. However, by doing so, the derivatives at the top and bottom interfaces of the thin-layers are not consistent with the tractions as calculated in the previous section, and therefore their degree of accuracy is not as good. In the reference [11], an alternative procedure for the calculation of the derivatives is proposed. The procedure is based on the definition of secondary interpolation functions, which are consistent with the stresses at the top and bottom interfaces of the thin-layer. In this work, the same procedure is used to define the vertical derivatives at the internal nodal interfaces.

Consider a thin-layer (of expansion nn), from which one knows the displacements $\bar{u}_{\alpha\beta j}$ at the $nn+1$ nodes, their horizontal derivatives ($\bar{u}_{\alpha\beta,xj}$ and $\bar{u}_{\alpha\beta,yj}$) and the consistent tractions ($\bar{p}_{\alpha\beta j}$) at the top and bottom interfaces, caused by a load in the direction β. The tractions at the upper surface relate to the internal stresses though $\tilde{\mathbf{p}}_1=\left\{\bar{\sigma}^{\mathrm{top}}_{xz\beta} \quad \bar{\sigma}^{\mathrm{top}}_{yz\beta} \quad -\mathrm{i}\bar{\sigma}^{\mathrm{top}}_{zz\beta}\right\}^{\mathrm{T}}$ and the tractions at the lower surface relate to the internal stresses through $\tilde{\mathbf{p}}_{nn+1}=-\left\{\bar{\sigma}^{\mathrm{bottom}}_{xz\beta} \quad \bar{\sigma}^{\mathrm{bottom}}_{yz\beta} \quad -\mathrm{i}\bar{\sigma}^{\mathrm{bottom}}_{zz\beta}\right\}^{\mathrm{T}}$. In its turn, the internal stresses and the derivatives of the displacements are related by

$$\begin{cases}\bar{\sigma}_{xz\beta}=G\left(\bar{u}_{x\beta,z}+\bar{u}_{z\beta,x}\right)\\ \bar{\sigma}_{yz\beta}=G\left(\bar{u}_{y\beta,z}+\bar{u}_{z\beta,y}\right)\\ \bar{\sigma}_{zz\beta}=\lambda\left(\bar{u}_{x\beta,x}+\bar{u}_{y\beta,y}\right)+\left(\lambda+2G\right)\bar{u}_{z\beta,z}\end{cases} \rightarrow \begin{cases}\bar{u}_{x\beta,z}=\left(\bar{\sigma}_{xz\beta}-G\bar{u}_{z\beta,x}\right)\big/G\\ \bar{u}_{y\beta,z}=\left(\bar{\sigma}_{yz\beta}-G\bar{u}_{z\beta,y}\right)\big/G\\ \bar{u}_{z\beta,z}=\left[\bar{\sigma}_{zz\beta}-\lambda\left(\bar{u}_{x\beta,x}+\bar{u}_{y\beta,y}\right)\right]\big/\left(\lambda+2G\right)\end{cases} \tag{8}$$

where G and λ are the Lamé constants. For each combination of α,β, the displacements and horizontal derivatives at each of the $nn+1$ nodal interfaces are known and therefore Eq. (8) can be used to determine the vertical derivatives at the upper and lower interfaces. Thus, with the $nn+3$ known quantities, one can make use of Hermitian interpolation and find a polynomial of degree $nn+2$ that approximates the vertical variation of the displacements, and with that polynomial, obtain the value of the vertical derivatives at the interior nodal interfaces. Knowing the vertical derivatives at all nodal interfaces and since the horizontal derivatives are also known, all the components of the internal stresses are ready to be found by employing the constitutive relations of the material.

3 DIRECT CALCULATION OF THE BEM COEFFICIENTS VIA THE TLM

In the context of the boundary element method and assuming constant expansion of the elements, coefficients of the form

$$\begin{aligned}H_{\alpha\beta}&=\int_\Gamma u_{\alpha\beta}\left(x,z,x_\xi,z_\xi,-k_y,\omega\right)\mathrm{d}\Gamma\\ Q_{\alpha\beta}&=\int_\Gamma p^{\mathbf{n}^*}_{\alpha\beta}\left(x,z,x_\xi,z_\xi,-k_y,\omega\right)\mathrm{d}\Gamma\end{aligned} \tag{9}$$

must be calculated. In Eq. (9), Γ represents the boundary of a boundary element, $\mathbf{x}_\xi = (x_\xi, y_\xi)$ represents a collocation point, $\mathbf{x} = (x, y)$ represents an observation point inside Γ and $p_{\alpha\beta}^{\mathbf{n}*}$ are the tractions at point $\mathbf{x}$ of the boundary Γ. In the following two sub-sections, it is shown how to calculate the coefficients $H_{\alpha\beta}$ and $Q_{\alpha\beta}$ for horizontal and for vertical boundaries using the TLM.

3.1 Horizontal boundaries

In this case, z and z_ξ are constant and consequently the integrals in Eq. (9) can be reduced to integrals in the horizontal coordinate, i.e.,

$$\begin{aligned} H_{\alpha\beta} &= \int_{-l/2}^{l/2} u_{\alpha\beta}^{mn}\left(x - x_\xi\right)\mathrm{d}x \\ Q_{\alpha\beta} &= \pm \int_{-l/2}^{l/2} \sigma_{\alpha z\beta}^{mn}\left(x - x_\xi\right)\mathrm{d}x \end{aligned} \tag{10}$$

being l the width of the boundary element, m the nodal depth of the boundary element, n the nodal depth of the collocation point and $\sigma_{\alpha z\beta}^{mn}$ internal stresses in horizontal planes, which correspond to the consistent nodal tractions as indicated in 2.3b.

The integrals in (10) can be replaced by the more convenient integrals

$$\begin{aligned} H_{\alpha\beta} &= -\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\bar{u}_{\alpha\beta}^{mn}\left(k_x\right)}{k_x}\left(\mathrm{e}^{\mathrm{i}\left(\frac{l}{2}+x_\xi\right)k_x} - \mathrm{e}^{\mathrm{i}\left(-\frac{l}{2}+x_\xi\right)}\right)\mathrm{d}k_x \\ Q_{\alpha\beta} &= \mp\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty} \frac{\bar{\sigma}_{\alpha z\beta}^{mn}\left(k_x\right)}{k_x}\left(\mathrm{e}^{\mathrm{i}\left(\frac{l}{2}+x_\xi\right)k_x} - \mathrm{e}^{\mathrm{i}\left(-\frac{l}{2}+x_\xi\right)}\right)\mathrm{d}k_x \end{aligned} \tag{11}$$

which in turn can be evaluated analytically since the integrands are known in closed form expressions. This way, for the evaluation of (11), one simply needs to evaluate integrals of the form

$$I_{ij}^{p} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} k_x^p\, K_{ij}\left(k_x\right)\mathrm{e}^{-\mathrm{i}k_x x}\,\mathrm{d}k_x \tag{12}$$

and then combine them properly according to Eq. (11) and sections 2.3a and 2.3b. For constant horizontal boundaries, the needed integrals are I_{ij}^{-1}, I_{ij}^{0} and I_{ij}^{1}. The integrals (12) are evaluated by means of contour integration, being the expressions for I_{ij}^{1} summarized in Appendix I. Expressions for I_{ij}^{-1} and I_{ij}^{0} can be found in [10].

It is important to note that some of the integrals $I_{ij}^{p}(x)$ contain factors of $\operatorname{sign}(x)$. Hence, when x results from some algebraic operation (e.g. $x = x_\xi - l/2$) and it is supposed to be zero, one needs to guarantee that x is indeed zero and not some residual value, as this will result in a erroneous evaluation of the integrals $I_{ij}^{p}(x)$.

As final comment concerning horizontal boundaries, the calculation of the coefficients $H_{\alpha\beta}$ involves only the components of the modal shapes at the elevation of the load and at the elevation of the receiver. In its turn, the calculation of the coefficients $Q_{\alpha\beta}$ involves the components of the eigenmodes at all nodes of the thin-layer that contains the boundary element.

Because the boundary elements are placed at the interface between two consecutive thin-layers, one needs to decide which thin-layer to consider: the one below or the one above. Though there is no difference when the collocation point does not belong to the boundary element, when it does belong, the consideration of one or the other results in different values for $Q_{\alpha\beta}$. In this work, if the outwards normal faces up, one considers the thin-layer below the boundary, while if the outwards normal faces down, one considers the thin-layer above the boundary. By following this procedure, one is excluding the collocation points from the domain, and so the singularity of the GFs is intrinsically removed.

3.2 Vertical boundaries

Vertical boundaries are defined by a constant horizontal coordinate x_{BE}. If one assumes that the collocation point is placed at the depth $z_\xi = z_l$ (l^{th} interface of the TLM model) and that the boundary element is placed between depths z_m and z_n (m^{th} and n^{th} interfaces of the TLM model), then the integrals (9) can be replaced by integrals of the form

$$\begin{aligned} H_{\alpha\beta} &= \sum_{k=m}^{n} \int u_{\alpha\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) N_k\left(z\right) \mathrm{d}z \\ Q_{\alpha\beta} &= \pm \sum_{k=m}^{n} \int \sigma_{\alpha x\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) N_k\left(z\right) \mathrm{d}z \end{aligned} \tag{13}$$

in which the integrands $u_{\alpha\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) N_k\left(z\right)$ and $\sigma_{\alpha x\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) N_k\left(z\right)$ represent the vertically interpolated displacement and traction fields, with $N_k\left(z\right)$ being the shape function associated to the k^{th} interface.

Because the values $u_{\alpha\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right)$ and $\sigma_{\alpha x\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right)$ are nodal values and therefore do not depend on the depth z, the expressions in Eq. (13) can be replaced by

$$\begin{aligned} H_{\alpha\beta} &= \sum_{k=m}^{n} u_{\alpha\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) \int N_k\left(z\right) \mathrm{d}z \\ Q_{\alpha\beta} &= \pm \sum_{k=m}^{n} \sigma_{\alpha x\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) \int N_k\left(z\right) \mathrm{d}z \end{aligned} \tag{14}$$

The integrals of the form $\int N_k\left(z\right)\mathrm{d}z$ are easy to evaluate since the functions $N_k\left(z\right)$ are polynomials. In their turn, the space domain fields $u_{\alpha\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right)$ and $\sigma_{\alpha x\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right)$ are calculated through the inverse Fourier transforms

$$\begin{aligned} u_{\alpha\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{u}_{\alpha\beta}^{kl}\left(k_x\right) \mathrm{e}^{-\mathrm{i}\left(x_{\mathrm{BE}} - x_\xi\right)} \, \mathrm{d}k_x \\ \bar{\sigma}_{\alpha x\beta}^{kl}\left(x_{\mathrm{BE}} - x_\xi\right) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{\sigma}_{\alpha x\beta}^{kl}\left(k_x\right) \mathrm{e}^{-\mathrm{i}\left(x_{\mathrm{BE}} - x_\xi\right)} \mathrm{d}k_x \end{aligned} \tag{15}$$

As for the case of horizontal boundary elements, the integrals in (20) can be evaluated analytically, and for that purpose one simply needs to combine the integrals I_{ij}^0, I_{ij}^1 and I_{ij}^2 according to the expressions of section 2.3a, 2.3b and 2.3c. The integrals I_{ij}^1 and I_{ij}^2 are given in Appendix I while the integrals I_{ij}^0 can be found in [10].

As a final note, since the displacements are interpolated in the vertical direction by means of polynomial functions, the singular behaviour of the GFs cannot be reproduced. Hence, when the collocation point is contained in the vertical boundary element, one must add 0.5 to the calculated value of $Q_{\alpha\beta}$. The value 0.5 results from the regularization process for smooth boundaries (a vertical boundary is a smooth boundary).

4 CONCLUSIONS

In this work, a 2.5D BEM procedure is developed based on the TLM Green's functions. For horizontal boundary elements, the BEM coefficients are directly calculated by modal superposition, rendering accurate results and accounting for the singularities of the GFs. For vertical boundary elements, the vertically interpolated GFs are integrated analytically but the GF singularities are not accounted for. To account for the singular behavior of the GF, one needs to add, a posteriori, 0.5 to the value of the calculated traction coefficients.

When compared to BEM procedures based on the analytical whole-space GF, the proposed procedure presents the advantage of being capable of considering horizontally layered domains with the same ease as homogeneous domains while avoiding the discretization of free surfaces and layer interfaces. When compared to BEM procedures based on GF obtained with transfer or stiffness matrices, the proposed method has the advantage of being capable of evaluating the inverse Fourier transform in closed form expressions, which yield more accurate results. Also, it becomes easier to use because the user only needs to discretize the layered domain in the vertical direction, a task that is far simpler than to define a proper wavenumber interval and to evaluate inverse Fourier transforms from k_x to x.

The proposed methodology considers only horizontal and vertical boundary elements. When the actual boundary presents inclined surfaces, such geometry can be achieved by filling the irregular volume with finite elements.

Due to the page limit of the document, it was not possible to include an example, but the procedure has been validated by comparison of results from different approaches. Also, expressions for linear and quadratic boundary elements have been obtained, but these are not shown here.

As a final remark, the TLM model must be compatible with the BEM mesh in such a way that:

1. The horizontal boundaries are placed at the interface between two different thin-layers and never inside a thin-layer;
2. The extremes of vertical boundaries correspond to interfaces separating thin-layers and never to the intermediate elevations within the thin-layers;
3. The nodes of vertical boundary elements must be located at the interface of thin-layers and never inside thin-layers;
4. It is not recommended that the horizontal boundary elements be smaller than the thickness of the thin-layers. Likewise, it is not recommended that the distance between vertical boundary elements at the same level be smaller than the thickness of the thin-layers.

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APPENDIX I – LIST OF INTEGRALS

I_{ij}^1 ($\mathrm{Im}\sqrt{k_j^2-k_y^2}<0$)

$$I_{1j}^1=\frac{\mathrm{sgn}(x)}{2\mathrm{i}}\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}$$

$$I_{2j}^1=\frac{k_y}{2\mathrm{i}k_j^2}\{\sqrt{k_j^2-k_y^2}\,\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}+\mathrm{i}\left|k_y\right|\mathrm{e}^{-\left|k_y x\right|}\}$$

$$I_{3j}^1=\frac{\mathrm{sgn}(x)}{2\mathrm{i}k_j^2}\{\left(k_j^2-k_y^2\right)\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}+k_y^2\,\mathrm{e}^{-\left|k_y x\right|}\}$$

$$I_{4j}^1=\frac{\mathrm{sgn}(x)}{2\mathrm{i}k_j^2}\{k_y^2\,\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}-k_y^2\,\mathrm{e}^{-\left|k_y x\right|}\}$$

$$I_{5j}^1=\frac{\sqrt{k_j^2-k_y^2}}{2\mathrm{i}k_j}\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}$$

$$I_{6j}^1=\frac{\mathrm{sgn}(x)k_y}{2\mathrm{i}k_j}\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}$$

I_{ij}^2 ($\mathrm{Im}\sqrt{k_j^2-k_y^2}<0$, $\mathrm{Im}\left(k_j^2-k_y^2\right)^{\frac{3}{2}}<0$)

$$I_{1j}^2=\frac{\sqrt{k_j^2-k_y^2}}{2\mathrm{i}}\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}$$

$$I_{2j}^2=\frac{\mathrm{sgn}(x)k_y}{2\mathrm{i}k_j^2}\{\left(k_j^2-k_y^2\right)\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}+k_y^2\,\mathrm{e}^{-\left|k_y x\right|}\}$$

$$I_{3j}^2=\frac{1}{2\mathrm{i}k_j^2}\{\left(k_j^2-k_y^2\right)^{3/2}\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}-\mathrm{i}\left|k_y\right|^3\mathrm{e}^{-\left|k_y x\right|}\}$$

$$I_{4j}^2=\frac{1}{2\mathrm{i}k_j^2}\{k_y^2\sqrt{k_j^2-k_y^2}\,\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}+\mathrm{i}\left|k_y\right|^3\mathrm{e}^{-\left|k_y x\right|}\}$$

$$I_{5j}^2=\frac{\mathrm{sgn}(x)\left(k_j^2-k_y^2\right)}{2\mathrm{i}k_j}\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}$$

$$I_{6j}^2=\frac{k_y\sqrt{k_j^2-k_y^2}}{2\mathrm{i}k_j}\mathrm{e}^{-\mathrm{i}|x|\sqrt{k_j^2-k_y^2}}$$

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