ALVARO F. M. AZEVEDO

Mecânica dos Sólidos - Colecção de Problemas Resolvidos

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MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

FOLHA 1 - CÁLCULO TENSORIAL

1 – Desenvolva as seguintes expressões

a) $a_{ij}x_j$

b) $\delta_{ij}t_{ju}$

c) $\delta_{ij}\delta_{ij}$

d) $\delta_{ik}\delta_{ku}t_{ul}$

2 – Prove a partir da definição que o produto contraído de dois vectores é um escalar.

3 – Sejam v_{ij}, c_{ij}, v'_{pq} e c'_{pq} os elementos nos referenciais *S* e *S'*, respectivamente, de dois tensores de 2ª ordem tais que $v_{ik}c_{kj} = \delta_{ij}$. Prove a partir da definição que $v'_{pr}c'_{rq} = \delta_{pq}$.

4 - Verifique a natureza tensorial da entidade com elementos t_{ij} num referencial *S* arbitrário que verifica a equação tensorial homogénea $t_{ij}v_ic_j = k$, em que v_i e c_j são as componentes em *S* de dois vectores arbitrários e *k* é um escalar.

5 - Prove a partir da definição que a soma dos elementos da diagonal principal de um tensor de 2ª ordem é um invariante.

6 - Sabendo que

$$\hat{e}_i \mid \hat{e}_j = \delta_{ij}$$
$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\hat{e}_k$$
$$\vec{a} = a_l\hat{e}_l$$

a) Calcule $\vec{a} \times \vec{b} \mid \vec{c}$

b) Demonstre que:

b.1) $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{a} \mid \vec{c} \cdot \vec{b} - \vec{a} \mid \vec{b} \cdot \vec{c}$

b.2) $\left(\vec{a} \times \vec{b}\right) \left| \left(\vec{c} \times \vec{d}\right) \right. = \left(\vec{a} \mid \vec{c}\right)\left(\vec{b} \mid \vec{d}\right) - \left(\vec{b} \mid \vec{c}\right)\left(\vec{a} \mid \vec{d}\right)$

7 - Considere dois referenciais ortonormados (S e S') sendo (x_1, x_2, x_3) os eixos de S e (x_1', x_2', x_3') os eixos de S'. Relativamente a estes dois referenciais sabe-se que:

- x_1' coincide com x_2
- x_3' faz um ângulo de 45º com x_1 e com x_3

Nota: estes dados referem-se aos semi-eixos positivos.

Sabendo que um ponto P possui coordenadas (5,2,4) em S, calcule as suas coordenadas em S'.

8 - Obtenha os elementos no referencial S do tensor anti-simétrico de 2ª ordem com elementos no referencial S'

$$t_{32}' = -1 \ \ ; \ \ t_{13}' = 2 \ \ ; \ \ t_{21}' = 0$$

sabendo que $\hat{e}_2 = \frac{1}{5}(3, ?, 4)_{S'}$; $\hat{e}_1 \,|\, \hat{e}_3' = 0$

9 - Determine o elemento t_{23} no referencial S de um tensor simétrico de 2ª ordem, sabendo que os seus elementos são

$$t_{11} = -t_{22} = 1 \ \ ; \ \ t_{12} = t_{33} = t_{13} = 0$$

e que no referencial S'

$$t_{33}' = -3 \quad ; \quad \hat{e}_3' = \frac{1}{3}(-1,2,2)_S$$

Resolução - FOLHA 1 - 1992/93 | R-1.1

Álvaro Azevedo

1 a) $a_{ij} v_j = a_{i1} v_1 + a_{i2} v_2 + a_{i3} v_3$

b) $\delta_{ij} t_{jm} = \delta_{i1} t_{1m} + \delta_{i2} t_{2m} + \delta_{i3} t_{3m} = t_{im}$

c) $\delta_{ij} \delta_{ij} = \delta_{i1} \delta_{i1} + \delta_{i2} \delta_{i2} + \delta_{i3} \delta_{i3} =$

$$= \underbrace{\delta_{11}\delta_{11}}_{=1} + \underbrace{\delta_{21}\delta_{21}}_{=0} + \underbrace{\delta_{31}\delta_{31}}_{=0} + \underbrace{\delta_{12}\delta_{12}}_{=0} + \underbrace{\delta_{22}\delta_{22}}_{=1} + \underbrace{\delta_{32}\delta_{32}}_{=0} + \underbrace{\delta_{13}\delta_{13}}_{=0} + \underbrace{\delta_{23}\delta_{23}}_{=0} + \underbrace{\delta_{33}\delta_{33}}_{=1} =$$

$= 1 + 1 + 1 = 3$

d) Recorrendo a b):

$\delta_{ik} \delta_{km} t_{ml} = \delta_{ik} t_{kl} = t_{il}$

e) Recorrendo a b)

$\delta_{ik} \delta_{km} \delta_{mn} t_{nl} = \delta_{ik} \delta_{km} t_{ml} = \delta_{ik} t_{kl} = t_{il}$

2 $u_i v_j = w_{ij} \xrightarrow[j=i]{\text{contracção}} w_{ii} = w_{11} + w_{22} + w_{33} = E$

$u_i v_j$: Produto de 2 vectores (tensores de 1ª ordem)

w_{ij}: Tensor de 2ª ordem

E: Escalar

R-1.2

(3) Dados $\begin{cases} v'_{pr} = a_{pi}\, a_{rj}\, v_{ij} \\ c'_{rq} = a_{rk}\, a_{ql}\, c_{kl} \\ v_{ik}\, c_{kj} = \delta_{ij} \end{cases}$

$$v'_{pr}\, c'_{rq} = a_{pi}\, a_{rj}\, v_{ij}\, a_{rk}\, a_{ql}\, c_{kl} =$$

$$= a_{pi}\, a_{ql}\, \underbrace{a_{rj}\, a_{rk}}_{\delta_{jk}}\, v_{ij}\, c_{kl} =$$

$$= a_{pi}\, a_{ql}\, \underbrace{\delta_{jk}\, v_{ij}}\, c_{kl} = a_{pi}\, a_{ql}\, \underbrace{v_{ik}\, c_{kl}}_{\delta_{il}\ (\text{dado})} =$$

$$= a_{pi}\, a_{ql}\, \delta_{il} = a_{pi}\, a_{qi} = \delta_{pq} \quad (\text{c.q.d.})$$

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(4) Dado $\{ t_{ij}\, v_i\, c_j = K$

$$t'_{pq}\, v'_p\, c'_q = a_{pi}\, a_{qj}\, t_{ij}\, a_{pl}\, v_l\, a_{qk}\, c_k =$$

$$= \underbrace{a_{pi}\, a_{pl}}_{\delta_{il}}\, \underbrace{a_{qj}\, a_{qk}}_{\delta_{jk}}\, t_{ij}\, v_l\, c_k =$$

$$= \delta_{il}\, \delta_{jk}\, t_{ij}\, v_l\, c_k = \delta_{il}\, t_{ik}\, v_l\, c_k =$$

$$= t_{lk}\, v_l\, c_k = \underset{(\text{escalar})}{K} = K'$$

$t_{ij}\, v_i\, c_j = K$

$t'_{pq}\, v'_p\, c'_q = K$

R-1.3

(5) $t_{ij} \to$ tensor de 2.ª ordem

$K = t_{ii} \to$ soma dos elementos da diagonal principal

$$t'_{pq} = a_{pi}\, a_{qj}\, t_{ij}$$

contracção $q = p$

$$\underbrace{t'_{pp}}_{K'} = \underbrace{a_{pi}\, a_{pj}}_{\delta_{ij}}\, t_{ij} = \delta_{ij}\, t_{ij} = t_{ii} = K$$

$$t'_{ii} = t_{ii}$$

$$K' = K$$

$K' = K$ (invariante)

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(6) a) $\vec{a} \times \vec{b} \,|\, \vec{c} = (a_i\, \hat{e}_i) \times (b_j\, \hat{e}_j) \,|\, (c_K\, \hat{e}_K) =$

$$= a_i\, b_j\, \varepsilon_{ijp}\, \hat{e}_p \,|\, c_K\, \hat{e}_K =$$

$$= a_i\, b_j\, c_K\, \varepsilon_{ijp}\, \underbrace{\hat{e}_p \,|\, \hat{e}_K}_{\delta_{pK}} =$$

$$= a_i\, b_j\, c_K\, \underbrace{\varepsilon_{ijp}\, \delta_{pK}}_{\varepsilon_{ijK}} =$$

$$= a_i\, b_j\, c_K\, \varepsilon_{ijK}$$

R − 1.4

b) b.1) $\vec{a} \times (\vec{b} \times \vec{c}) \overset{?}{=} \vec{a}|\vec{c} \cdot \vec{b} - \vec{a}|\vec{b} \cdot \vec{c}$

$$\vec{a} \times (\vec{b} \times \vec{c}) = a_i \hat{e}_i \times (b_j \hat{e}_j \times c_k \hat{e}_k) =$$

$$= a_i b_j c_k \, \hat{e}_i \times (\hat{e}_j \times \hat{e}_k) =$$

$$= a_i b_j c_k \, \hat{e}_i \times \varepsilon_{jkp} \hat{e}_p =$$

$$= a_i b_j c_k \, \varepsilon_{jkp} \, \hat{e}_i \times \hat{e}_p =$$

$$= a_i b_j c_k \, \varepsilon_{jkp} \, \varepsilon_{ipq} \hat{e}_q =$$

$$= a_i b_j c_k \, \varepsilon_{jkp} \, \varepsilon_{pqi} \hat{e}_q =$$

$$= a_i b_j c_k \left(\delta_{jq} \delta_{ki} - \delta_{ji} \delta_{kq} \right) \hat{e}_q =$$

$$= a_i b_j c_k \delta_{jq} \delta_{ki} \hat{e}_q - a_i b_j c_k \delta_{ji} \delta_{kq} \hat{e}_q =$$

$$b_q \qquad c_i \qquad\qquad b_i \qquad c_q$$

$$= \underbrace{a_i c_i}_{\vec{a}|\vec{c}} \underbrace{b_q \hat{e}_q}_{\vec{b}} - \underbrace{a_i b_i}_{\vec{a}|\vec{b}} \underbrace{c_q \hat{e}_q}_{\vec{c}} =$$

$$= \vec{a}|\vec{c} \cdot \vec{b} - \vec{a}|\vec{b} \cdot \vec{c}$$ (c.q.d.)

R-1.5

b.2) $\varepsilon_{ijk}\,\varepsilon_{ijk} \overset{?}{=} 6$

Sabendo que:

$$\varepsilon_{ijk}\,\varepsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\varepsilon_{ijk}\,\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

Por contracção $q = j$:

$$\varepsilon_{ijk}\,\varepsilon_{pjk} = \delta_{ip}\underbrace{\delta_{jj}}_{=3} - \underbrace{\delta_{ij}\delta_{jp}}_{=\delta_{ip}}$$

$$\varepsilon_{ijk}\,\varepsilon_{pjk} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip} \quad (*)$$

Por contracção $p = i$:

$$\varepsilon_{ijk}\,\varepsilon_{ijk} = 2\underbrace{\delta_{ii}}_{=3} = 6 \quad \text{(c.q.d.)}$$

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b.3) De $(*)$ em b.2):

$$\varepsilon_{ijk}\,\varepsilon_{pjk} = 2\delta_{ip}$$

$$\varepsilon_{jki}\,\varepsilon_{jkp} = 2\delta_{ip}$$

$$\left.\begin{array}{l} j \to i \\ k \to j \\ i \to k \\ p \to l \end{array}\right\} \quad \varepsilon_{ijk}\,\varepsilon_{ijl} = 2\delta_{kl}$$

(c.q.d.)

b. 4) $(\vec{a}\times\vec{b})|(\vec{c}\times\vec{d}) \overset{?}{=} (\vec{a}|\vec{c})\cdot(\vec{b}|\vec{d}) - (\vec{b}|\vec{c})\cdot(\vec{a}|\vec{d})$ R-1.6

$$(\vec{a}\times\vec{b})|(\vec{c}\times\vec{d}) = \left[(a_i\,\hat{e}_i)\times(b_j\,\hat{e}_j)\right] \Big| \left[(c_p\,\hat{e}_p)\times(d_q\,\hat{e}_q)\right] =$$

$$= (a_i\, b_j\, \varepsilon_{ijk}\, \hat{e}_k) \,\big|\, (c_p\, d_q\, \varepsilon_{pqr}\, \hat{e}_r) =$$

$$= a_i\, b_j\, c_p\, d_q\, \varepsilon_{ijk}\, \varepsilon_{pqr} \underbrace{(\hat{e}_k|\hat{e}_r)}_{=\delta_{kr}} =$$

$$= a_i\, b_j\, c_p\, d_q\, \varepsilon_{ijk}\, \varepsilon_{pqk} =$$

$$= a_i\, b_j\, c_p\, d_q\, (\delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp}) =$$

$$= a_i \underbrace{c_p\,\delta_{ip}}_{=c_i} \underbrace{\delta_{jq}\, d_q}_{=d_j} b_j - b_j \underbrace{d_q\,\delta_{iq}}_{=d_i} \underbrace{\delta_{jp}\, c_p}_{=c_j} a_i =$$

$$= a_i\, c_i\, b_j\, d_j - b_j\, c_j\, a_i\, d_i =$$

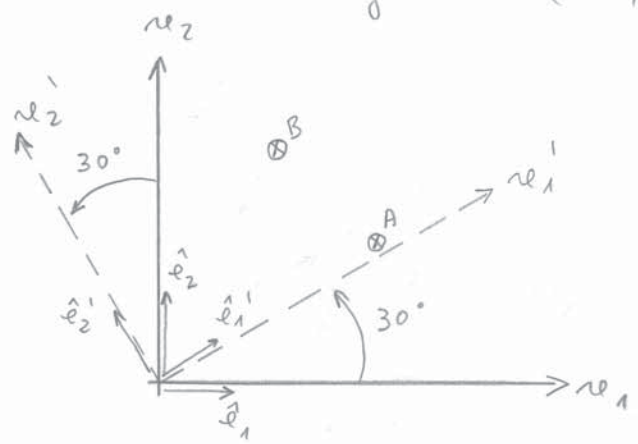
$$= (\vec{a}|\vec{c})\cdot(\vec{b}|\vec{d}) - (\vec{b}|\vec{c})\cdot(\vec{a}|\vec{d})$$

(c.q.d)

R - 1.7

7

$$a_{ij} = \cos(x_i', x_j) = \hat{e}_i' \mid \hat{e}_j$$



$$a_{11} = \cos(x_1', x_1) = \cos 30° = \frac{\sqrt{3}}{2}$$

$$a_{12} = \cos(x_1', x_2) = \cos 60° = \frac{1}{2}$$

$$a_{21} = \cos(x_2', x_1) = \cos 120° = -\frac{1}{2}$$

$$a_{22} = \cos(x_2', x_2) = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\underset{\sim}{A} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

a)

$$\underset{\sim}{x}_A = (3,2) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\underset{\sim}{x}_A' = \underset{\sim}{A}\,\underset{\sim}{x}_A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\sqrt{3} + 1 \\ -\frac{3}{2} + \sqrt{3} \end{bmatrix} = \begin{bmatrix} 3.598 \\ 0.232 \end{bmatrix}$$

b)

$$\underset{\sim}{x}_B' = (3,2) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\underset{\sim}{x}_B = \underset{\sim}{A}^T \underset{\sim}{x}_B' = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}\sqrt{3} - 1 \\ \frac{3}{2} + \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1.598 \\ 3.232 \end{bmatrix}$$

Ver resolução alternativa na página R-1.11

R-1.8

8)

$$[\underset{\sim}{t}'] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} \quad \text{(anti simétrico)}$$

$$\underset{\sim}{v}' = \underset{\sim}{A}\,\underset{\sim}{v} \longrightarrow \underset{\sim}{A} = \begin{bmatrix} \boxed{\hat{e}_1'} \\ \boxed{\hat{e}_2'} \\ \boxed{\hat{e}_3'} \end{bmatrix}_S$$

$$\underset{\sim}{v} = \underset{\sim}{B}\,\underset{\sim}{v}' \longrightarrow \underset{\sim}{B} = \begin{bmatrix} \boxed{\hat{e}_1} \\ \boxed{\hat{e}_2} \\ \boxed{\hat{e}_3} \end{bmatrix}_{S'}$$

$$\underset{\sim}{B} = \underset{\sim}{A}^T \Leftrightarrow \underset{\sim}{A} = \underset{\sim}{B}^T$$

$$\underset{\sim}{t}' = \underset{\sim}{A}\,\underset{\sim}{t}\,\underset{\sim}{A}^T$$

multiplicando ambos os membros por $\underset{\sim}{A}^{-1} = \underset{\sim}{A}^T$ vem:

$$\underset{\sim}{A}^T\underset{\sim}{t}' = \underset{\sim}{t}\,\underset{\sim}{A}^T$$

de um modo semelhante:

$$\underset{\sim}{A}^T\underset{\sim}{t}'\underset{\sim}{A} = \underset{\sim}{t}$$

$$A = \begin{bmatrix} \boxed{\hat{e}_1} & \boxed{\hat{e}_2} & \boxed{\hat{e}_3} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & 3/5 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 4/5 & a_{33} \end{bmatrix}$$

porque $\hat{e}_1 | \hat{e}_3' = \hat{e}_3' | \hat{e}_1 = a_{31} = 0$

R - 1.9

Sabendo que $\| \hat{e}_2 \| = 1$

$$\left(\frac{3}{5}\right)^2 + a_{22}^2 + \left(\frac{4}{5}\right)^2 = 1 \Rightarrow a_{22} = 0$$

Sabendo que $\hat{e}_1 | \hat{e}_2 = 0$ (ortogonais)

$$a_{11} \times \frac{3}{5} + a_{21} \times \underset{\overset{\shortparallel}{0}}{a_{22}} + 0 \times \frac{4}{5} = 0 \Rightarrow a_{11} = 0$$

Sabendo que $\| \hat{e}_1 \| = 1$

$$\underset{\overset{\shortparallel}{0}}{a_{11}^2} + a_{21}^2 + 0 = 1 \Rightarrow a_{21} = \pm 1$$

Arbitrando $a_{21} = 1$ tem-se

$$\hat{e}_1 = (0, 1, 0)$$

$$\hat{e}_2 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$$

$$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 = \begin{vmatrix} \hat{e}_1' & \hat{e}_2' & \hat{e}_3' \\ 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{vmatrix} = \left(\frac{4}{5}, 0, -\frac{3}{5}\right)_{S'}$$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \end{bmatrix}$$

R - 1.10

$$\underset{\sim}{t} = \underset{\sim}{A}^T \underset{\sim}{t}' \underset{\sim}{A}$$

$$\underset{\sim}{t} = \begin{bmatrix} 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \\ 4/5 & 0 & -3/5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -8/5 & -4/5 & 6/5 \\ 6/5 & 3/5 & 8/5 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 4/5 & -3/5 \\ -4/5 & 0 & -2 \\ 3/5 & 2 & 0 \end{bmatrix}}_{\text{também anti simétrica}}$$

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$$\underset{\sim}{t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & t_{23} \\ 0 & t_{32} & 0 \end{bmatrix} \quad (t_{23} = t_{32}) \Rightarrow \underset{\sim}{t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & \kappa \\ 0 & \kappa & 0 \end{bmatrix}$$

$$\text{Em } S' \begin{cases} t'_{33} = -3 \\ \hat{e}'_3 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)_S = (a_{31}, a_{32}, a_{33}) \end{cases}$$

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$$t'_{pq} = a_{pi} a_{qj} t_{ij}$$

$$-3 = t'_{33} = a_{3i} a_{3j} t_{ij} = a_{31} a_{3j} t_{1j} + a_{32} a_{3j} t_{2j} + a_{33} a_{3j} t_{3j} =$$

$$= a_{31} a_{31} \overset{=1}{t_{11}} + a_{31} a_{32} \overset{=0}{t_{12}} + a_{31} a_{33} \overset{=0}{t_{13}} + a_{32} a_{31} \overset{=0}{t_{21}} + a_{32} a_{32} \overset{=-1}{t_{22}} + a_{32} a_{33} \overset{=\kappa}{t_{23}} +$$

$$+ a_{33} a_{31} \overset{=0}{t_{31}} + a_{33} a_{32} \overset{=\kappa}{t_{32}} + a_{33} a_{33} \overset{=0}{t_{33}} = a_{31}^2 - a_{32}^2 + a_{32} a_{33} \kappa + a_{33} a_{32} \kappa =$$

$$= \frac{1}{9} - \frac{4}{9} + \frac{4}{9}\kappa + \frac{4}{9}\kappa \iff -3 = -\frac{3}{9} + \frac{8}{9}\kappa \iff \kappa = -3$$

(8) $\boxed{\text{Nova resolução}}$ (1997/9/27)

<u>Enunciado</u>: obtenha os elementos no referencial S'' do tensor anti-simétrico de 2ª ordem com os seguintes elementos no referencial S'

$$\underset{\sim}{t}' = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$$

Sabe-se que $\hat{e}_2'' = \left(\frac{3}{5}, a, \frac{4}{5}\right)_{S'}$ e que $\hat{e}_1'' | \hat{e}_3' = 0$

$\hat{e}_2'' | \hat{e}_1' = \frac{3}{5}$ $\qquad$ $\hat{e}_2'' | \hat{e}_2' = a$ $\qquad$ $\hat{e}_2'' | \hat{e}_3' = \frac{4}{5}$

$\hat{e}_1'' | \hat{e}_3' = 0$

$$\underset{\sim}{A} = \begin{bmatrix} \hat{e}_1''|\hat{e}_1' & \hat{e}_1''|\hat{e}_2' & \hat{e}_1''|\hat{e}_3' \\ \hat{e}_2''|\hat{e}_1' & \hat{e}_2''|\hat{e}_2' & \hat{e}_2''|\hat{e}_3' \\ \hat{e}_3''|\hat{e}_1' & \hat{e}_3''|\hat{e}_2' & \hat{e}_3''|\hat{e}_3' \end{bmatrix} = \begin{bmatrix} b & c & 0 \\ 3/5 & a & 4/5 \\ d & e & f \end{bmatrix}$$

$$\|\hat{e}_2''\| = 1 \Rightarrow \left(\frac{3}{5}\right)^2 + a^2 + \left(\frac{4}{5}\right)^2 = 1 \Rightarrow \frac{9}{25} + a^2 + \frac{16}{25} = 1 \Rightarrow a = 0$$

$$\underset{\sim}{A} = \begin{bmatrix} b & c & 0 \\ 3/5 & 0 & 4/5 \\ d & e & f \end{bmatrix}$$

Ortogonalidade:

$a_{ij} a_{kj} = \delta_{ik} \Rightarrow \underset{\sim}{A}\,\underset{\sim}{A}^T = \underset{\sim}{I}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(a_{11}, a_{12}, a_{13}) | (a_{21}, a_{22}, a_{23}) = 0$

$(b, c, 0) \; | \; (3/5, 0, 4/5) = 0$

Considerando S' como referencial geral, tem-se **R-1.12**

$$(b, c, 0) | \left(\frac{3}{5}, 0, \frac{4}{5}\right) = 0$$

$$\frac{3}{5} b = 0 \Rightarrow b = 0$$

$c = \pm 1 \rightsquigarrow$ adopta-se $c = 1$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \\ d & e & f \end{bmatrix}$$

$$(d, e, f) = (0, 1, 0) \wedge \left(\frac{3}{5}, 0, \frac{4}{5}\right) = \begin{vmatrix} \cdot & \cdot & \cdot \\ 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{vmatrix} = \left(\frac{4}{5}, 0, -\frac{3}{5}\right)$$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \\ 4/5 & 0 & -3/5 \end{bmatrix}$$

$$\underset{\sim}{t}'' = \underset{\sim}{A}\, \underset{\sim}{t}'\, \underset{\sim}{A}^T$$

$$\underset{\sim}{t}'' = \begin{bmatrix} 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \\ 4/5 & 0 & -3/5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \\ 4/5 & 0 & -3/5 \end{bmatrix} \begin{bmatrix} 0 & 8/5 & -6/5 \\ 0 & 4/5 & -3/5 \\ -1 & -6/5 & -8/5 \end{bmatrix} = \begin{bmatrix} 0 & 4/5 & -3/5 \\ -4/5 & 0 & -2 \\ 3/5 & 2 & 0 \end{bmatrix}$$

MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

FOLHA 2 - ESTADO DE TENSÃO

1 – O campo de tensões num meio contínuo é caracterizado pelo tensor

$$\tau_{ij} = \begin{bmatrix} x_1^2 x_2 & \left(1-x_2^2\right)x_1 & 0 \\ \left(1-x_2^2\right)x_1 & \left(x_2^3-3x_2\right)/3 & 0 \\ 0 & 0 & 2x_3^2 \end{bmatrix}$$

a) Determine as forças mássicas que deverão estar aplicadas a esse meio, de forma a satisfazer o equilíbrio em qualquer ponto.

b) Determine as tensões no ponto $P\left(a\ ,0\ ,2\sqrt{a}\ \right)$ e numa faceta igualmente inclinada relativamente aos semi-eixos positivos das coordenadas.

2 – Dado o estado de tensão

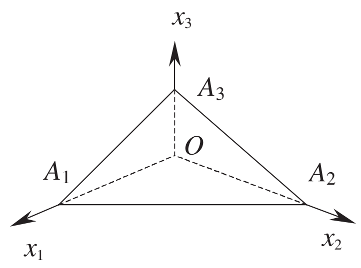
$$\tau_{ij} = \frac{1}{3}\begin{bmatrix} 16 & 4 & 2 \\ 4 & 16 & 2 \\ 2 & 2 & 22 \end{bmatrix} (\text{MPa})$$

expresso no referencial $S \equiv \left(0,\hat{e}_1,\hat{e}_2,\hat{e}_3\right)$, determine as componentes do mesmo estado de tensão no referencial $S' \equiv \left(0,\hat{e}_1',\hat{e}_2',\hat{e}_3'\right)$, sabendo que

$$\begin{Bmatrix} x_1' \\ x_2' \\ x_3' \end{Bmatrix} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

3 – O tetraedro representado encontra-se em equilíbrio sujeito a um estado de tensão uniforme, definido pelo tensor das tensões τ_{ij} com componentes em $(0, x_1, x_2, x_3)$:

$$\tau_{ij} = \begin{bmatrix} 20 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & -20 \end{bmatrix} (MPa)$$



$$\overline{OA_1} = \overline{OA_2} = 3\sqrt{2}\,cm$$
$$\overline{OA_3} = 4\,cm$$

a) Determine a resultante das forças que actuam na face (A_1, A_2, A_3).

b) Determine a orientação das facetas para as quais é máximo o valor da tensão normal.

c) Determine no referencial dado a orientação das facetas para as quais é máximo o valor absoluto da tensão tangencial.

4 – Relativamente ao estado de tensão definido por:

$$\tau_{11} = \tau_{22} = -\frac{2}{3}\,MPa \quad ; \quad \tau_{33} = \frac{4}{3}\,MPa$$

$$\tau_{12} = \frac{4}{3}\,MPa \quad ; \quad \tau_{23} = \tau_{31} = \frac{2}{3}\,MPa$$

Calcule o valor das tensões principais e caracterize as suas direcções de actuação.

5 – Considere o estado de tensão cujas tensões principais são:

$$\sigma_I = 12\,MPa$$
$$\sigma_{II} = 2\,MPa$$
$$\sigma_{III} = -4\,MPa$$

Em relação a um triedro directo formado pelas direcções principais, determine:

a) A orientação das facetas para as quais τ assume o valor máximo.

b) A orientação das facetas para as quais $\sigma = 0$.

Resolução - FOLHA 2 - 1992/93 | R-2.1

(1) a) $f_{m_i} + \dfrac{\partial \sigma_{ij}}{\partial x_j} = 0$

$\dfrac{\partial \sigma_{11}}{\partial x_1} = 2x_1x_2$	$\dfrac{\partial \sigma_{12}}{\partial x_2} = -2x_1x_2$	$\dfrac{\partial \sigma_{13}}{\partial x_3} = 0$
$\dfrac{\partial \sigma_{21}}{\partial x_1} = 1-x_2^2$	$\dfrac{\partial \sigma_{22}}{\partial x_2} = x_2^2-1$	$\dfrac{\partial \sigma_{23}}{\partial x_3} = 0$
$\dfrac{\partial \sigma_{31}}{\partial x_1} = 0$	$\dfrac{\partial \sigma_{32}}{\partial x_2} = 0$	$\dfrac{\partial \sigma_{33}}{\partial x_3} = 4x_3$

$$\left.\begin{aligned} f_{m_1} + (2x_1x_2 - 2x_1x_2 + 0) = 0 &\Rightarrow f_{m_1} = 0 \\ f_{m_2} + (1-x_2^2 + x_2^2 - 1 + 0) = 0 &\Rightarrow f_{m_2} = 0 \\ f_{m_3} + (0 + 0 + 4x_3) = 0 &\Rightarrow f_{m_3} = -4x_3 \end{aligned}\right\} \vec{f}_m = \begin{bmatrix} 0 \\ 0 \\ -4x_3 \end{bmatrix}$$

---

b) Substituindo $P = (a, 0, 2\sqrt{a})$ em σ_{ij}:

$$\sigma_{ij} = \begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 8a \end{bmatrix} \qquad \hat{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\vec{t}_P^{(\hat{n})} = [\sigma_{ij}]\,\hat{n} = \begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 8a \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} a/\sqrt{3} \\ a/\sqrt{3} \\ 8a/\sqrt{3} \end{bmatrix}$$

$$\vec{\sigma} = \vec{t}_P^{(\hat{n})} \cdot \hat{n}\,\hat{n} = \left(\frac{a}{3} + \frac{a}{3} + \frac{8a}{3}\right)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{10a}{3\sqrt{3}}, \frac{10a}{3\sqrt{3}}, \frac{10a}{3\sqrt{3}}\right)$$

$$\vec{\tau} = \vec{t}_P^{(\hat{n})} - \vec{\sigma} = \left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{8a}{\sqrt{3}}\right) - \left(\frac{10a}{3\sqrt{3}}, \frac{10a}{3\sqrt{3}}, \frac{10a}{3\sqrt{3}}\right) = \left(-\frac{7a}{3\sqrt{3}}, -\frac{7a}{3\sqrt{3}}, \frac{14a}{3\sqrt{3}}\right)$$

2

$$\underset{\sim}{A} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

$$\underset{\sim}{x}' = \underset{\sim}{A}\,\underset{\sim}{x}$$

$$\underset{\sim}{\sigma}' = \underset{\sim}{A}\,\underset{\sim}{\sigma}\,\underset{\sim}{A}^T$$

$$A = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 2 \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & 0 \end{bmatrix} \qquad \sigma = \frac{1}{3} \begin{bmatrix} 16 & 4 & 2 \\ 4 & 16 & 2 \\ 2 & 2 & 22 \end{bmatrix}$$

$$A\sigma = \frac{1}{3\sqrt{6}} \begin{bmatrix} 24 & 24 & 48 \\ -18\sqrt{2} & -18\sqrt{2} & 18\sqrt{2} \\ 12\sqrt{3} & -12\sqrt{3} & 0 \end{bmatrix}$$

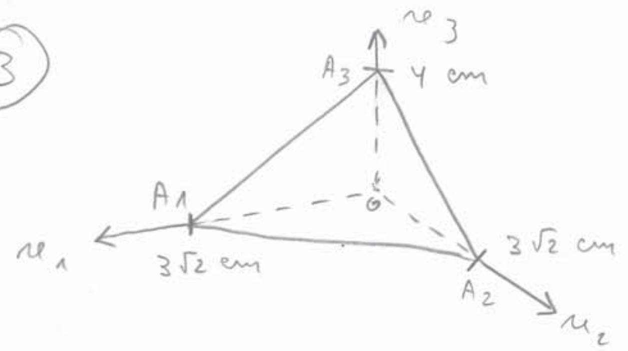
$$A\sigma A^T = \frac{1}{3\sqrt{6}}\frac{1}{\sqrt{6}} \begin{bmatrix} 24 & 24 & 48 \\ -18\sqrt{2} & -18\sqrt{2} & 18\sqrt{2} \\ 12\sqrt{3} & -12\sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{2} & \sqrt{3} \\ 1 & -\sqrt{2} & -\sqrt{3} \\ 2 & \sqrt{2} & 0 \end{bmatrix} =$$

$$A\sigma A^T = \frac{1}{18} \begin{bmatrix} 144 & 0 & 0 \\ 0 & 108 & 0 \\ 0 & 0 & 72 \end{bmatrix} = \underbrace{\begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{\text{tensor principal}} (MPa)$$

$$\begin{cases} \sigma_1 = 8 \\ \sigma_2 = 6 \\ \sigma_3 = 4 \end{cases}$$

→ a matriz $\underset{\sim}{A}$ é constituída pelos versores das direcções principais

③ R-2.3



Equação do plano que passa pelos 3 pontos:

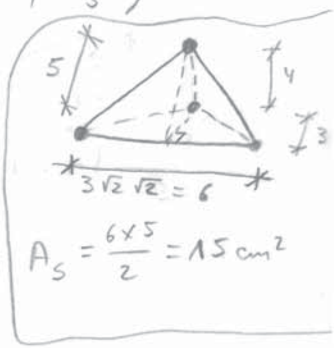
$$\frac{x_1}{3\sqrt{2}} + \frac{x_2}{3\sqrt{2}} + \frac{x_3}{4} = 1$$

$$\vec{n} = \left(\frac{1}{3\sqrt{2}},\ \frac{1}{3\sqrt{2}},\ \frac{1}{4}\right) \qquad \|\vec{n}\| = \sqrt{\frac{1}{18} + \frac{1}{18} + \frac{1}{16}} = \frac{5}{12}$$

$$\hat{n} = \frac{\vec{n}}{\|\vec{n}\|} = \left(\frac{12}{15\sqrt{2}},\ \frac{12}{15\sqrt{2}},\ \frac{3}{5}\right) = \left(\frac{2\sqrt{2}}{5},\ \frac{2\sqrt{2}}{5},\ \frac{3}{5}\right)$$

a)

$$\vec{t}^{(\hat{n})} = \sigma\,\hat{n} = \begin{bmatrix} 20 & 10 & 0 \\ 10 & 20 & 0 \\ 0 & 0 & -20 \end{bmatrix} \begin{bmatrix} \frac{2\sqrt{2}}{5} \\ \frac{2\sqrt{2}}{5} \\ 3/5 \end{bmatrix} = \begin{bmatrix} 12\sqrt{2} \\ 12\sqrt{2} \\ -12 \end{bmatrix} (MPa)$$



$$A_{s_i} = A_s\, \hat{n} \,|\, \hat{e}_i \Rightarrow A_{s_3} = \text{Área}(0, A_1, A_2) = A_s \underbrace{\hat{n} \,|\, \hat{e}_3}_{3/5}$$

$$A_{s_3} = \frac{3\sqrt{2}\cdot 3\sqrt{2}}{2} = 9\ cm^2 \Rightarrow 9 = A_s\,\frac{3}{5} \Rightarrow A_s = 15\ cm^2$$

$$A_s = 15\,E{-4}\ m^2$$

$$\vec{R} = t^{(\hat{n})}\cdot A_s = \begin{bmatrix} 12\sqrt{2} \\ 12\sqrt{2} \\ -12 \end{bmatrix} 15\,E{-4} = \begin{bmatrix} 0.018\sqrt{2} \\ 0.018\sqrt{2} \\ -0.018 \end{bmatrix} MN$$

$$\vec{R} = 0.018\left(\sqrt{2},\ \sqrt{2},\ -1\right) MN$$

R-2.4

(3) b) Calcular as tensões principais para determinar a maior:

$$\begin{vmatrix} (20-\sigma) & 10 & 0 \\ 10 & (20-\sigma) & 0 \\ 0 & 0 & (-20-\sigma) \end{vmatrix} = 0 \Leftrightarrow (-20-\sigma)\left[(20-\sigma)^2-100\right]$$

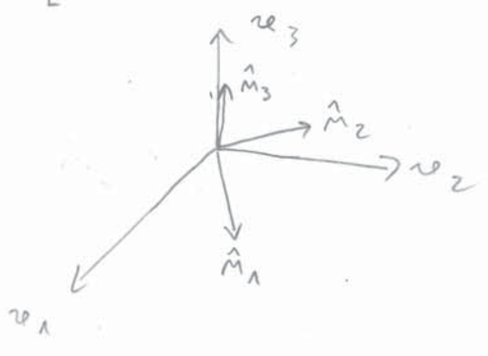
Soluções $\begin{cases} \sigma = -20 \\ \sigma = 30 \\ \sigma = 10 \end{cases}$

$$\begin{cases} \sigma_1 = 30 \leftarrow \text{Maior} \\ \sigma_2 = 10 \\ \sigma_3 = -20 \end{cases} \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

σ_1 $$\begin{bmatrix} -10 & 10 & 0 \\ 10 & -10 & 0 \\ 0 & 0 & -50 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -10n_1 + 10n_2 = 0 \\ 10n_1 - 10n_2 = 0 \\ -50\, n_3 = 0 \end{cases} \begin{cases} n_1 = 1/\sqrt{2} \\ n_2 = 1/\sqrt{2} \\ n_3 = 0 \end{cases}$$

σ_2 $$\begin{bmatrix} 10 & 10 & 0 \\ 10 & 10 & 0 \\ 0 & 0 & -30 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 10n_1 + 10n_2 = 0 \\ 10n_1 + 10n_2 = 0 \\ -30\, n_3 = 0 \end{cases} \begin{cases} n_1 = -1/\sqrt{2} \\ n_2 = 1/\sqrt{2} \\ n_3 = 0 \end{cases}$$

σ_3 $$\begin{bmatrix} 40 & 10 & 0 \\ 10 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 40n_1 + 10n_2 = 0 \\ 10n_1 + 40n_2 = 0 \\ 0 = 0 \end{cases} \begin{cases} n_1 = 0 \\ n_2 = 0 \\ n_3 = 1 \end{cases}$$



É necessário que $\hat{n}_1 \times \hat{n}_2 = \hat{n}_3$

$$\begin{vmatrix} \cdot & \cdot & \cdot \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{vmatrix} = (0, 0, 1) \quad (OK)$$

③ c) R-2.5

$$\tau_{máx} = \frac{\sigma_1 - \sigma_3}{2} = \frac{30-(-20)}{2} = 25\ MPa$$

Ocorre nas facetas A e B paralelas a x_2' e fazendo um ângulo de 45° com x_1' e x_3'

$(x_1', x_2', x_3') \rightarrow$ referencial principal

$x' = A\,x$

$x = A^T x'$

$$A = \begin{bmatrix} \boxed{\hat{n}_1} \\ \boxed{\hat{n}_2} \\ \boxed{\hat{n}_3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Em S': $\hat{n}_A' = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$; $\hat{n}_B' = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$\hat{n}_A = A^T \hat{n}_A' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\hat{n}_B = A^T \hat{n}_B' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

Versores em S das facetas em que ocorre $\tau_{máx}$

4 a) $\sigma_1 = 12$; $\sigma_2 = 2$; $\sigma_3 = -4$ $\boxed{R-2.6}$

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = 8 \text{ MPa}$$

$$\hat{n}_A = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$\hat{n}_B = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

---

b) $\sigma = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = 0$

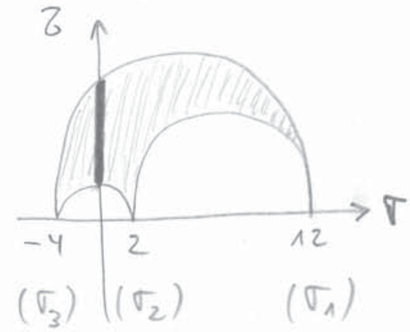
$$12 n_1^2 + 2 n_2^2 - 4 n_3^2 = 0 \Rightarrow n_3^2 = 3n_1^2 + \frac{n_2^2}{2}$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow n_1^2 + n_2^2 + 3n_1^2 + n_2^2/2 = 1 \Rightarrow$$

$$\Rightarrow 4n_1^2 + 3n_2^2/2 = 1 \Rightarrow n_2^2 = \frac{2}{3}(1 - 4n_1^2) \Rightarrow n_2^2 = \frac{2}{3} - \frac{8}{3} n_1^2$$

$$n_3^2 = 3n_1^2 + \frac{1}{3} - \frac{4}{3} n_1^2 \Rightarrow n_3^2 = \frac{5}{3} n_1^2 + \frac{1}{3}$$

$$\begin{cases} n_1 \rightarrow \text{variável} \\ n_2 = \pm\sqrt{\dfrac{-8n_1^2 + 2}{3}} \Rightarrow -8n_1^2 + 2 \geq 0 \Rightarrow n_1^2 \leq 1/4 \\ n_3 = \pm\sqrt{\dfrac{5n_1^2 + 1}{3}} \Rightarrow 5n_1^2 + 1 \geq 0 \Rightarrow n_1^2 \geq -1/5 \rightarrow \text{é sempre} \end{cases}$$



$$\begin{cases} -\dfrac{1}{2} \leq n_1 \leq \dfrac{1}{2} \\ n_2 = \pm\sqrt{\dfrac{-8n_1^2 + 2}{3}} \\ n_3 = \pm\sqrt{\dfrac{5n_1^2 + 1}{3}} \end{cases}$$

R - 2.7

(5)

$$\tau = \begin{bmatrix} -2/3 & 4/3 & 2/3 \\ 4/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 4/3 \end{bmatrix}$$

$$\left| \underset{\sim}{\tau} - \sigma \underset{\sim}{I} \right| = 0 \Rightarrow \begin{vmatrix} \left(-\frac{2}{3}-\sigma\right) & \frac{4}{3} & \frac{2}{3} \\ \frac{4}{3} & \left(-\frac{2}{3}-\sigma\right) & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \left(\frac{4}{3}-\sigma\right) \end{vmatrix} = 0$$

$$\left(-\frac{2}{3}-\sigma\right)\left[\left(-\frac{2}{3}-\sigma\right)\left(\frac{4}{3}-\sigma\right)-\frac{4}{9}\right]-\frac{4}{3}\left[\frac{4}{3}\left(\frac{4}{3}-\sigma\right)-\frac{4}{9}\right]+\frac{2}{3}\left[\frac{8}{9}-\frac{2}{3}\left(-\frac{2}{3}-\sigma\right)\right]=0$$

$$\left(-\frac{2}{3}-\sigma\right)\left[-\frac{8}{9}+\frac{2}{3}\sigma-\frac{4}{3}\sigma+\sigma^2-\frac{4}{9}\right]-\frac{4}{3}\left[\frac{16}{9}-\frac{4}{3}\sigma-\frac{4}{9}\right]+\frac{2}{3}\left[\frac{8}{9}+\frac{4}{9}+\frac{2}{3}\sigma\right]=0$$

$$\left(-\frac{2}{3}-\sigma\right)\left[-\frac{4}{3}-\frac{2}{3}\sigma+\sigma^2\right]-\frac{4}{3}\left[\frac{4}{3}-\frac{4}{3}\sigma\right]+\frac{2}{3}\left[\frac{4}{3}+\frac{2}{3}\sigma\right]=0$$

$$\frac{8}{9}+\frac{4}{9}\sigma-\frac{2}{3}\sigma^2+\frac{4}{3}\sigma+\frac{2}{3}\sigma^2-\sigma^3-\frac{16}{9}+\frac{16}{9}\sigma+\frac{8}{9}+\frac{4}{9}\sigma=0$$

$$4\sigma-\sigma^3=0 \Rightarrow \sigma^3-4\sigma=0 \Rightarrow \sigma^2-4=0 \begin{cases} \sigma_1 = 2 \\ \sigma_2 = 0 \\ \sigma_3 = -2 \end{cases}$$

Confirmação: $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$$I_1 = -\frac{2}{3}-\frac{2}{3}+\frac{4}{3} = 0$$

$$I_2 = \frac{4}{9}-\frac{16}{9}-\frac{8}{9}-\frac{4}{9}-\frac{8}{9}-\frac{4}{9} = -4$$

$$I_3 = -\frac{2}{3}\left(-\frac{8}{9}-\frac{4}{9}\right)-\frac{4}{3}\left(\frac{16}{9}-\frac{4}{9}\right)+\frac{2}{3}\left(\frac{8}{9}+\frac{4}{9}\right) = \frac{24}{27}-\frac{48}{27}+\frac{24}{27} = 0$$

OK

Direcções principais

R-2.8

$$\begin{bmatrix} -8/3 & 4/3 & 2/3 \\ 4/3 & -8/3 & 2/3 \\ 2/3 & 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left(n_1^2 + n_2^2 + n_3^2 = 1 \right)$$

$$\begin{cases} -8n_1 + 4n_2 + 2n_3 = 0 \\ 4n_1 - 8n_2 + 2n_3 = 0 \\ 2n_1 + 2n_2 - 2n_3 = 0 \end{cases} \quad \begin{cases} 2n_3 = 8n_1 - 4n_2 \\ 4n_1 - 8n_2 + 8n_1 - 4n_2 = 0 \\ 2n_1 + 2n_2 - 8n_1 + 4n_2 = 0 \end{cases} \quad \begin{cases} n_3 = 4n_1 - 2n_2 \\ 12n_1 = 12n_2 \\ \cancel{+6n_1 = 6n_2} \end{cases}$$

$$n_1^2 + n_1^2 + (4n_1 - 2n_1)^2 = 1 \Rightarrow 2n_1^2 + 4n_1^2 = 1 \Rightarrow n_1 = \frac{1}{\sqrt{6}}$$

$$n_1 = n_2 = \frac{1}{\sqrt{6}} \ ; \ n_3 = \frac{2}{\sqrt{6}} \qquad \boxed{\hat{n} = \frac{1}{\sqrt{6}}(1, 1, 2)} \quad (\sigma_1)$$

$$\begin{bmatrix} -2/3 & 4/3 & 2/3 \\ 4/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 4/3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left(n_1^2 + n_2^2 + n_3^2 = 1 \right)$$

$$\begin{cases} -2n_1 + 4n_2 + 2n_3 = 0 \\ 4n_1 - 2n_2 + 2n_3 = 0 \\ 2n_1 + 2n_2 + 4n_3 = 0 \end{cases} \quad \begin{cases} 2n_3 = 2n_1 - 4n_2 \\ 4n_1 - 2n_2 + 2n_1 - 4n_2 = 0 \\ 2n_1 + 2n_2 + 4n_1 - 8n_2 = 0 \end{cases} \quad \begin{cases} n_3 = n_1 - 2n_2 \\ 6n_1 = 6n_2 \\ \cancel{6n_1 = 6n_2} \end{cases}$$

$$n_1^2 + n_1^2 + (n_1 - 2n_1)^2 = 1 \Rightarrow 2n_1^2 + n_1^2 = 1 \Rightarrow n_1 = \frac{1}{\sqrt{3}}$$

$$n_1 = n_2 = \frac{1}{\sqrt{3}} \ ; \ n_3 = -\frac{1}{\sqrt{3}} \qquad \boxed{\hat{n} = \frac{1}{\sqrt{3}}(1, 1, -1)} \quad (\sigma_2)$$

$$\hat{n}_1 \times \hat{n}_2 = \begin{vmatrix} \cdot & \cdot & \cdot \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{vmatrix} = \begin{cases} n_1 = -\frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} = -\frac{3}{3\sqrt{2}} = -\frac{1}{\sqrt{2}} \\ n_2 = \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \\ n_3 = \frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}} = 0 \end{cases} \qquad \boxed{\hat{n} = \frac{1}{\sqrt{2}}(-1, 1, 0)} \ (\sigma_3$$

$\boxed{R-2.9}$

(6) No referencial das tensões principais: $\hat{n} = \frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$

$$\sigma_{oct} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_1 \frac{1}{3} + \sigma_2 \frac{1}{3} + \sigma_3 \frac{1}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \Rightarrow \boxed{\sigma_{oct} = \frac{I_1}{3}} \text{ (c.q.d.)}$$

$$\tau_{oct}^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - \sigma_{oct}^2 = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{3} - \left(\frac{I_1}{3}\right)^2$$

Sabendo que:

$$I_1^2 = (\sigma_1 + \sigma_2 + \sigma_3)^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\underbrace{(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)}_{I_2}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = I_1^2 - 2 I_2$$

Substituindo atrás:

$$\tau_{oct}^2 = \frac{I_1^2 - 2I_2}{3} - \frac{I_1^2}{9} = \frac{2I_1^2 - 6I_2}{9}$$

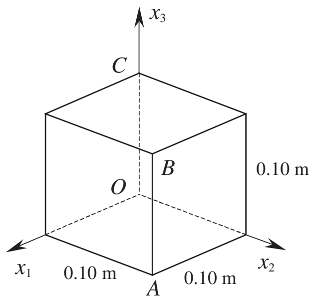
$$\boxed{\tau_{oct} = \frac{\sqrt{2}}{3}\sqrt{I_1^2 - 3I_2}} \text{ (c.q.d.)}$$

MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

FOLHA 3 - ESTADO DE TENSÃO

1 – Relativamente a um estado de tensão uniforme caracterizado pelo tensor das tensões:

$$[\tau]=\begin{bmatrix} -10 & 10 & 0 \\ 10 & 0 & -20 \\ 0 & -20 & 20 \end{bmatrix}(\text{MPa})$$



a) Determine a resultante das tensões em cada face do cubo e na face OABC.

b) Determine as facetas em que a tensão tangencial é máxima.

2 – Considere um estado de tensão num ponto de cujo tensor se conhecem as seguintes componentes: $\tau_{12} = 2$ MPa; $\tau_{22} = -2$ MPa; $\tau_{13} = \tau_{23} = 0$. Sabe-se ainda que uma das tensões principais é nula e que as outras duas apresentam o mesmo valor absoluto, sendo uma positiva e a outra negativa. Calcule as componentes do tensor das tensões.

3 – Relativamente ao estado de tensão num ponto, sabe-se que a direcção principal correspondente à maior tensão principal é definida, num referencial $S \equiv (0, x_1, x_2, x_3)$ pelo versor $\hat{n}_I = \frac{1}{3}(2,-2,1)_S$. Na superfície cuja normal é $\frac{1}{\sqrt{2}}(1,0,1)_S$, a componente tangencial da tensão tem o seu valor máximo de 300 MPa e a correspondente componente normal vale 500 MPa. O invariante linear das tensões vale 1500 MPa.

Determine:

a) As tensões principais e as direcções principais de tensão.

b) Os elementos do tensor das tensões no referencial *S*.

Solução do problema 3 - a)

$$\sigma_I = 800\,MPa\,;\hat{n}_I = \frac{1}{3}(2,-2,1);\sigma_{II} = 500\,MPa\,;\hat{n}_{II} = \frac{1}{3}(2,1,-2);\sigma_{III} = 200\,MPa\,;\hat{n}_{III} = \frac{1}{3}(1,2,2)$$

Solução do problema 3 - b) $\quad [\tau]_S = \begin{bmatrix} 600 & -200 & 0 \\ -200 & 500 & 200 \\ 0 & 200 & 400 \end{bmatrix} (\mathrm{MPa})$

4 – Relativamente a um estado de tensão num ponto, sabe-se que:

i) Existe um elemento de superfície onde não actua nenhuma tensão;

ii) Uma direcção principal de tensão é definida por um versor cujas componentes no referencial S são $\frac{1}{\sqrt{5}}(2,1,0)_S$;

iii) O elemento de superfície cujo versor da normal é $\frac{1}{3}(1,2,2)$ está sujeito a uma tensão de (40,0,0) MPa.

Determine:

a) As tensões principais e as direcções principais de tensão;

b) Os elementos do tensor das tensões no referencial *S*.

Solução do problema 4 - a)

$$\sigma_I = 60\,MPa\,;\hat{n}_I = \frac{1}{\sqrt{5}}(2,1,0);\sigma_{II} = 0;\hat{n}_{II} = (0,0,1);\sigma_{III} = -40\,MPa\,;\hat{n}_{III} = \frac{1}{\sqrt{5}}(1,-2,0)$$

Solução do problema 4 - b) $\quad [\tau] = \begin{bmatrix} 40 & 40 & 0 \\ 40 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\mathrm{MPa})$

Resolução - FOLHA 3 - 1992/93

R-3.1

① Dados:

$$\begin{cases} \text{Direcção correspondente a } \sigma_1 \rightarrow \hat{n}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \\ \tau_{máx} = 300 \quad \text{para } \hat{n} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\ \text{Para } \tau_{máx}: \sigma = 500 \\ I_1 = 1500 \end{cases}$$

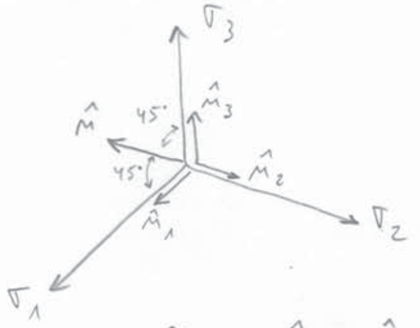
a) Tensões e direcções principais?

$$\tau_{máx} = \frac{\sigma_1 - \sigma_3}{2} = 300 \Rightarrow \sigma_1 - \sigma_3 = 600$$

$$\frac{\sigma_1 + \sigma_3}{2} = 500 \Rightarrow \sigma_1 + \sigma_3 = 1000$$

$$\begin{cases} \sigma_1 = 800 \\ \sigma_3 = 200 \end{cases}$$

$$\sigma_1 + \sigma_2 + \sigma_3 = 1500 \Rightarrow 800 + \sigma_2 + 200 = 1500 \Rightarrow \sigma_2 = 500$$



$\hat{n}$ está no plano $(\hat{n}_1, \hat{n}_3)$

$\hat{n}_2 \perp$ ao plano $(\hat{n}, \hat{n}_1)$

$$\vec{n}_2 = \hat{n} \times \hat{n}_1 = \begin{vmatrix} \cdot & \cdot & \cdot \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 2/3 & -2/3 & 1/3 \end{vmatrix} = \left(\frac{2}{3\sqrt{2}}, \frac{2}{3\sqrt{2}} - \frac{1}{3\sqrt{2}}, -\frac{2}{3\sqrt{2}}\right)$$

$$\|\vec{n}_2\| = \sqrt{\left(\frac{2}{3\sqrt{2}}\right)^2 + \left(\frac{1}{3\sqrt{2}}\right)^2 + \left(\frac{2}{3\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}$$

$$\hat{n}_2 = \frac{\vec{n}_2}{\|\vec{n}_2\|} = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$

$$\hat{n}_3 = \hat{n}_1 \times \hat{n}_2 = \begin{vmatrix} \cdot & \cdot & \cdot \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{vmatrix} = \left(\frac{4}{9} - \frac{1}{9}, \frac{2}{9} + \frac{4}{9}, \frac{2}{9} + \frac{4}{9}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$\boxed{\begin{aligned} \hat{n}_1 &= \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \\ \hat{n}_2 &= \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \\ \hat{n}_3 &= \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \end{aligned}}$$

R-3.2

1 b) Tensor das tensões?

$$\underset{\sim}{\tau}' = \underset{\sim}{A}\,\underset{\sim}{\tau}\,\underset{\sim}{A}^T \Rightarrow \underset{\sim}{\tau} = \underset{\sim}{A}^T\underset{\sim}{\tau}'\underset{\sim}{A}$$

$$A = \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$$

$$A^T\tau' = \frac{1}{3}\times 100 \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 16 & 10 & 2 \\ -16 & 5 & 4 \\ 8 & -10 & 4 \end{bmatrix} \times \frac{100}{3}$$

$$A^T\tau' A = \frac{100}{3}\times\frac{1}{3}\begin{bmatrix} 16 & 10 & 2 \\ -16 & 5 & 4 \\ 8 & -10 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} = \frac{100}{9}\begin{bmatrix} 54 & -18 & 0 \\ -18 & 45 & -18 \\ 0 & -18 & 36 \end{bmatrix}$$

$$\tau = \begin{bmatrix} 600 & -200 & 0 \\ -200 & 500 & -200 \\ 0 & -200 & 400 \end{bmatrix}$$

R-3.3

② a)

$$\vec{t}_{(\hat{m}_A)} = \underset{\sim}{\sigma}\, \hat{m}_A$$

$$\hat{m}_B^T \vec{t}_{(\hat{m}_A)} = \hat{m}_B^T \underset{\sim}{\sigma}\, \hat{m}_A$$

$$\vec{t}_{(\hat{m}_A)}^{\,T} \hat{m}_B = \hat{m}_A^T \underset{\sim}{\sigma}\, \hat{m}_B$$

$$(\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T)$$

$$\vec{t}_{(\hat{m}_B)} = \underset{\sim}{\sigma}\, \hat{m}_B$$

$$\hat{m}_A^T \vec{t}_{(\hat{m}_B)} = \hat{m}_A^T \underset{\sim}{\sigma}\, \hat{m}_B$$

$$\vec{t}_{(\hat{m}_A)}^{\,T} \hat{m}_B = \hat{m}_A^T \vec{t}_{(\hat{m}_B)}$$

$$(1\times 3)\ (3\times 1)\quad (1\times 3)\ (3\times 1)$$

$$\vec{t}_{(\hat{m}_A)} \,\Big|\, \hat{m}_B = \vec{t}_{(\hat{m}_B)} \,\Big|\, \hat{m}_A$$

Facetas principais → A, B, C

Faceta não principal → F

Faceta A : $\hat{m}_A = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)_S$; $\vec{t}_{(\hat{m}_A)} = \left(\frac{2\sigma_A}{\sqrt{5}}, \frac{\sigma_A}{\sqrt{5}}, 0\right)$

Faceta B : $\hat{m}_B = \left(m_{B1}, m_{B2}, m_{B3}\right)_S$; $\vec{t}_{(\hat{m}_B)} = (0, 0, 0)$

Faceta F : $\hat{m}_F = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)_S$; $\vec{t}_{(\hat{m}_F)} = (40, 0, 0)$

(A ↔ F) $\left(\frac{2\sigma_A}{\sqrt{5}}, \frac{\sigma_A}{\sqrt{5}}, 0\right) \Big| \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = (40, 0, 0) \Big| \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$

$$\frac{2\sigma_A}{3\sqrt{5}} + \frac{2\sigma_A}{3\sqrt{5}} = \frac{80}{\sqrt{5}} \Rightarrow \sigma_A = 60$$

$$\vec{t}_{(\hat{m}_A)} = \left(\frac{120}{\sqrt{5}}, \frac{60}{\sqrt{5}}, 0\right)$$

R - 3.4

$$B \leftrightarrow F \quad (0,0,0) \,\Big|\, \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = (40,0,0) \,\Big|\, (m_{B1}, m_{B2}, m_{B3})$$

$$0 = 40\, m_{B_1} \Rightarrow m_{B_1} = 0$$

$$A \leftrightarrow B \quad \left(\frac{120}{\sqrt{5}}, \frac{60}{\sqrt{5}}, 0\right) \,\Big|\, (0, m_{B2}, m_{B3}) = (0,0,0) \,\Big|\, \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$\frac{60}{\sqrt{5}} m_{B_2} = 0 \Rightarrow m_{B_2} = 0$$

$$m_{B_1}^2 + m_{B_2}^2 + m_{B_3}^2 = 1 \Rightarrow m_{B_3} = \pm 1 \nearrow m_{B_3} = 1 \text{ (arbitrado)}$$

$$\hat{m}_A = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right) \qquad \hat{m}_B = (0,0,1)$$

$$\hat{m}_C = \hat{m}_A \times \hat{m}_B = \begin{vmatrix} \bullet & \bullet & \bullet \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$$

$$C \leftrightarrow F \quad \left(\frac{\sigma_C}{\sqrt{5}}, -\frac{2\sigma_C}{\sqrt{5}}, 0\right) \,\Big|\, \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = (40,0,0) \,\Big|\, \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$$

$$\frac{\sigma_C}{3\sqrt{5}} - \frac{4\sigma_C}{3\sqrt{5}} = \frac{40}{\sqrt{5}} \Rightarrow \sigma_C = -40$$

$$\sigma_1 = \sigma_A = 60 \quad ; \quad \hat{m}_1 = \hat{m}_A = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$\sigma_2 = \sigma_B = 0 \quad ; \quad \hat{m}_2 = \hat{m}_B = (0,0,1)$$

$$\sigma_3 = \sigma_C = -40 \quad ; \quad \hat{m}_3 = \hat{m}_C = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$$

② b) $\underset{\sim}{\sigma} = \underset{\sim}{A}^T \underset{\sim}{\sigma}' \underset{\sim}{A}$ $\qquad \underset{\sim}{A} = \begin{bmatrix} \boxed{\quad \hat{n}_1 \quad} \\ \boxed{\quad \hat{n}_2 \quad} \\ \boxed{\quad \hat{n}_3 \quad} \end{bmatrix}$ $\qquad$ R - 3.5

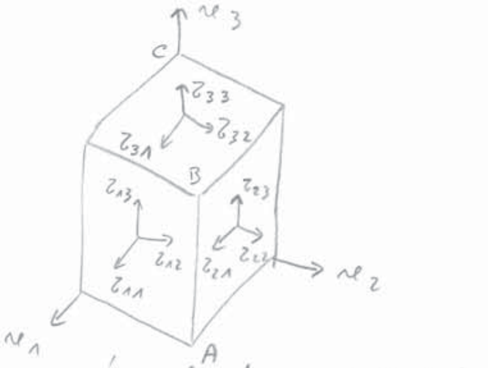
$$\underset{\sim}{A} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & -2/\sqrt{5} & 0 \end{bmatrix} \qquad \underset{\sim}{\sigma}' = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$\underset{\sim}{A}^T \underset{\sim}{\sigma}' = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 60 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -40 \end{bmatrix} = \begin{bmatrix} 120/\sqrt{5} & 0 & -40/\sqrt{5} \\ 60/\sqrt{5} & 0 & 80/\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{A}^T \underset{\sim}{\sigma}' \underset{\sim}{A} = \begin{bmatrix} 120/\sqrt{5} & 0 & -40/\sqrt{5} \\ 60/\sqrt{5} & 0 & 80/\sqrt{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & -2/\sqrt{5} & 0 \end{bmatrix} = \begin{bmatrix} 40 & 40 & 0 \\ 40 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

---

③ a) $\underset{\sim}{\sigma} = \begin{bmatrix} \boxed{-10 \quad 10 \quad 0} \\ \boxed{10 \quad 0 \quad -20} \\ \boxed{0 \quad -20 \quad 20} \end{bmatrix}$ (MPa)



$\hat{n} = (1,0,0) \Rightarrow \vec{t} = (-10, 10, 0)$

$\vec{R} = \Omega \vec{t} = 0.01(-10,10,0) = (-0.1, 0.1, 0)$

Ω = Área da face $= 0.1 \times 0.1$

$\Omega = 0.01\ m^2$

etc.

Em OABC: $\hat{n} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ $\qquad \vec{t} = \underset{\sim}{\sigma}\,\hat{n} = \begin{bmatrix} -10 & 10 & 0 \\ 10 & 0 & -20 \\ 0 & -20 & 20 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$

$\vec{t} = \left(-\frac{20}{\sqrt{2}}, \frac{10}{\sqrt{2}}, \frac{20}{\sqrt{2}}\right)$; $\vec{R} = \Omega \vec{t} \Rightarrow \vec{R} = 0.1 \times 0.1\sqrt{2}\left(-\frac{20}{\sqrt{2}}, \frac{10}{\sqrt{2}}, \frac{20}{\sqrt{2}}\right)$

$\vec{R} = (-0.2,\ 0.1,\ 0.2)$ (MN)

R-3.6

3 b)

$$\underset{\sim}{\tau} = \begin{bmatrix} -10 & 10 & 0 \\ 10 & 0 & -20 \\ 0 & -20 & 20 \end{bmatrix}$$

$$I_1 = -10 + 0 + 20 = 10$$

$$I_2 = \begin{vmatrix} -10 & 10 \\ 10 & 0 \end{vmatrix} + \begin{vmatrix} -10 & 0 \\ 0 & 20 \end{vmatrix} + \begin{vmatrix} 0 & -20 \\ -20 & 20 \end{vmatrix} = -100 - 200 - 400 = -700$$

$$I_3 = \begin{vmatrix} -10 & 10 & 0 \\ 10 & 0 & -20 \\ 0 & -20 & 20 \end{vmatrix} = -10(-400) - 10(200) = 2000$$

$$\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0$$

$$\sigma^3 - 10\sigma^2 - 700\sigma - 2000 = 0 \quad \begin{cases} \sigma_1 = 33.0278 \\ \sigma_2 = -3.0278 \\ \sigma_3 = -20 \end{cases}$$

Calcular as direcções principais $\begin{cases} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{cases}$

$$n' = A\, n$$

n': referencial principal

n: referencial inicial

$$A = \begin{bmatrix} \boxed{\quad \hat{n}_1 \quad} \\ \boxed{\quad \hat{n}_2 \quad} \\ \boxed{\quad \hat{n}_3 \quad} \end{bmatrix}$$

$$\boxed{n = A^T n'}$$

Facetas de $\tau_{máx}$ em S' $\begin{cases} \hat{n}' = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \\ \hat{n}' = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \end{cases}$

Facetas de $\tau_{máx}$ em S $\begin{cases} \hat{n} = [A]^T \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \dots \\ \hat{n} = [A]^T \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \dots \end{cases}$

④

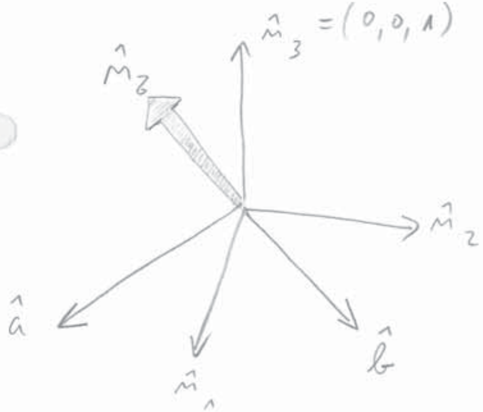
R-3.7

$\tau(\hat{a}) = 1.5 \text{ MPa}$

$\sigma(\hat{b}) = 7.5 \text{ MPa}$

$\hat{n}_{\tau_{max}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right)$ → no referencial $(\hat{a}\hat{b}\hat{n}_3)$

$\tau_{max} = 6 \text{ MPa}$



$$\vec{n}_2 = \hat{n}_3 \wedge \hat{n}_\tau = \begin{vmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 1/\sqrt{2} \end{vmatrix}$$

$$\vec{n}_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0 \right)$$

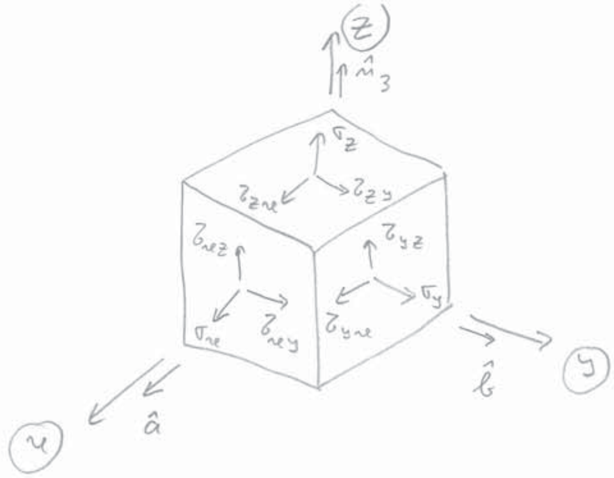
$$\| \vec{n}_2 \| = \frac{1}{\sqrt{2}}$$

$$\hat{n}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\hat{n}_1 = \hat{n}_2 \wedge \hat{n}_3 = \begin{vmatrix} \cdot & \cdot & \cdot \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

$$\left\{ \begin{array}{l} \hat{n}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\ \hat{n}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \\ \hat{n}_3 = (0, 0, 1) \end{array} \right. \longrightarrow \text{no referencial } (\hat{a}\hat{b}\hat{n}_3)$$

R-3.8



$$\begin{cases} \sigma_x = \sigma_a \\ \sigma_y = \sigma_b = 7.5 \\ \sigma_z = \sigma_3 \end{cases}$$

$$\tau_{zx} = \tau_{zy} = \tau_{xz} = \tau_{yz} = 0$$

$$\tau_{xy} = \tau_{yx} = \tau_a = \tau_b = 1.5$$

$$\underset{\sim}{\tau} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_a & \tau_a & 0 \\ \tau_a & \sigma_b & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_a & 1.5 & 0 \\ 1.5 & 7.5 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$\underset{\sim}{\tau}\, \hat{n}_1 = \sigma_1 \hat{n}_1 \Rightarrow \begin{bmatrix} \sigma_a & 1.5 & 0 \\ 1.5 & 7.5 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_1/\sqrt{2} \\ \sigma_1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\begin{cases} \sigma_a/\sqrt{2} + 1.5/\sqrt{2} = \sigma_1/\sqrt{2} \\ 1.5/\sqrt{2} + 7.5/\sqrt{2} = \sigma_1/\sqrt{2} \end{cases} \quad \begin{cases} \boxed{\sigma_a = 7.5} \\ \boxed{\sigma_1 = 9} \end{cases}$$

$$\underset{\sim}{\tau}\, \hat{n}_2 = \sigma_2 \hat{n}_2 \Rightarrow \begin{bmatrix} 7.5 & 1.5 & 0 \\ 1.5 & 7.5 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sigma_2/\sqrt{2} \\ \sigma_2/\sqrt{2} \\ 0 \end{bmatrix} \Rightarrow \boxed{\sigma_2 = 6}$$

$$\tau_{máx} = \frac{\sigma_1 - \sigma_3}{2} \Rightarrow 6 = \frac{9 - \sigma_3}{2} \Rightarrow \boxed{\sigma_3 = -3}$$

R-3.9

a) $\sigma_a = 7.5 \text{ MPa}$

b) $\begin{cases} \hat{n}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ \hat{n}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \\ \hat{n}_3 = (0, 0, 1) \end{cases}$ $\begin{cases} \sigma_1 = 9 \text{ MPa} \\ \sigma_2 = 6 \text{ MPa} \\ \sigma_3 = -3 \text{ MPa} \end{cases}$

c) No referencial das tensões principais:

$$\underset{\sim}{\sigma} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\hat{n}_{oct} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

versor de uma das facetas octaédricas

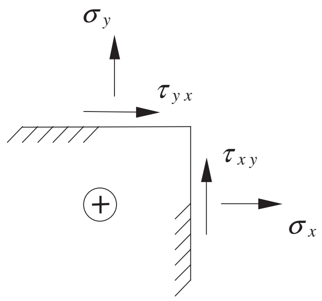
$$\vec{t} = \underset{\sim}{\sigma}\,\hat{n} = \left(\frac{9}{\sqrt{3}}, \frac{6}{\sqrt{3}}, -\frac{3}{\sqrt{3}}\right) \text{ (MPa)}$$

numa faceta octaédrica

MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

FOLHA 4 - ESTADO PLANO DE TENSÃO

1 – De um estado plano de tensão conhecem-se as tensões em duas facetas ortogonais:



$$\sigma_x = 100\,MPa$$
$$\sigma_y = 60\,MPa$$
$$\tau_{xy} = -20\,MPa$$

Determine a grandeza e direcção das tensões principais.

Solução: $\sigma_I = 108.28$ MPa ; $\sigma_{II} = 51.72$ MPa ; $\alpha_I = -22.5^o$; $\alpha_{II} = 67.5^o$

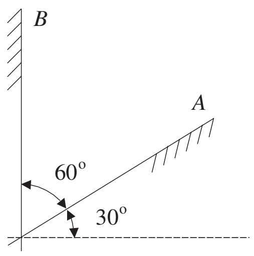
2 – As tensões principais num estado plano de tensão valem 500 MPa e –100 MPa, ocorrendo a primeira na faceta *A* representada na figura.

Determine:

a) As componentes normal e tangencial da tensão que actua na faceta *B*.

b) As orientações das facetas onde a tensão é puramente tangencial.

c) As orientações das facetas para as quais a tensão tangencial vale 200 MPa e as componentes normais das tensões nessas facetas.

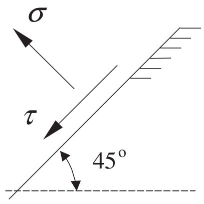


Solução a) $\sigma_B = 50$ MPa ; $\tau_B = -259.81$ MPa

Solução b) $\alpha = 5.905^o$; 54.096^o

Solução c) $\alpha = -80.904^o; 50.905^o$; $\sigma'_1 = 423.61$ MPa ; $\sigma'_2 = -23.61$ MPa

3 – De um estado plano de tensão conhecem-se o invariante linear das tensões, que vale 200 MPa e as componentes normal e tangencial da tensão que actua na faceta representada na figura.



σ = 400 MPa

τ = 200 MPa

Determine:

a) As tensões principais.

b) As orientações das facetas onde ocorrem os valores máximos e mínimos da componente tangencial da tensão.

Solução a) $\sigma_I = 460.56$ MPa ; $\sigma_{II} = -260.56$ MPa

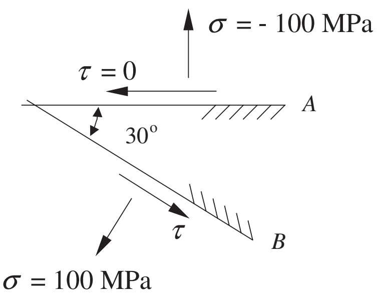
Solução b) $\alpha = 16.845°; -73.155°$

4 – Num estado plano de tensão conhece-se a tensão que actua na faceta *A* e a componente normal da tensão que actua na faceta *B*.

Determine:

a) As tensões principais.

b) A componente tangencial da tensão que actua na faceta *B*.



Solução a) $\sigma_I = 700$ MPa ; $\sigma_{II} = -100$ MPa

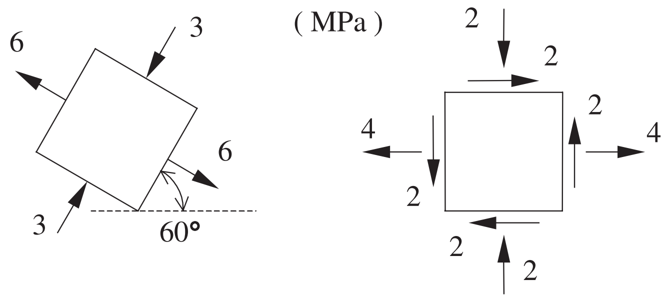
Solução b) $\tau_B = -346.41$ MPa

5 – Um ponto P de um sólido está submetido a um estado de tensão que é a soma dos dois estados de tensão que se indicam na figura.

Calcule:

a) O valor das tensões principais e a orientação das facetas principais.

b) As componentes da tensão que actua nos planos que fazem 30º com a faceta onde actua a tensão principal máxima.



Solução a) $\sigma_I = 8.08$ MPa ; $\alpha_I = -9.934º$; $\sigma_{II} = -3.08$ MPa ; $\alpha_{II} = 80.066º$

Solução b) $\sigma = 5.29$ MPa ; $\tau = -4.83$ MPa

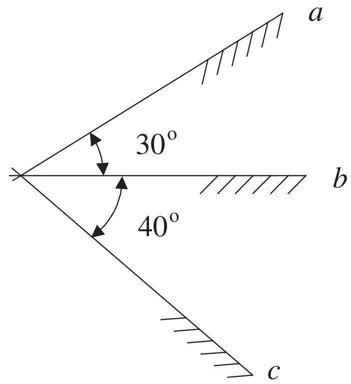
6 – Considere as facetas a, b e c em que:

$\sigma_a = 100$ MPa

$\sigma_b = 200$ MPa

$\sigma_c = 300$ MPa

Calcule τ_a, τ_b e τ_c.



Solução:

$\tau_a = -63.412$ MPa

$\tau_b = -109.793$ MPa

$\tau_c = -9.382$ MPa

Resolução - FOLHA 4 - 1992/93

R - 4.1

1 a) $\sigma_1 = 500 \quad (\alpha = -60°)$ $\qquad \sigma_x = ?$, $\tau_{xy} = ?$ $\}\ \alpha = 0°$

$\sigma_2 = -100$

$$500 = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\,\mathrm{sen}(2\alpha) \qquad (\alpha = -60)$$

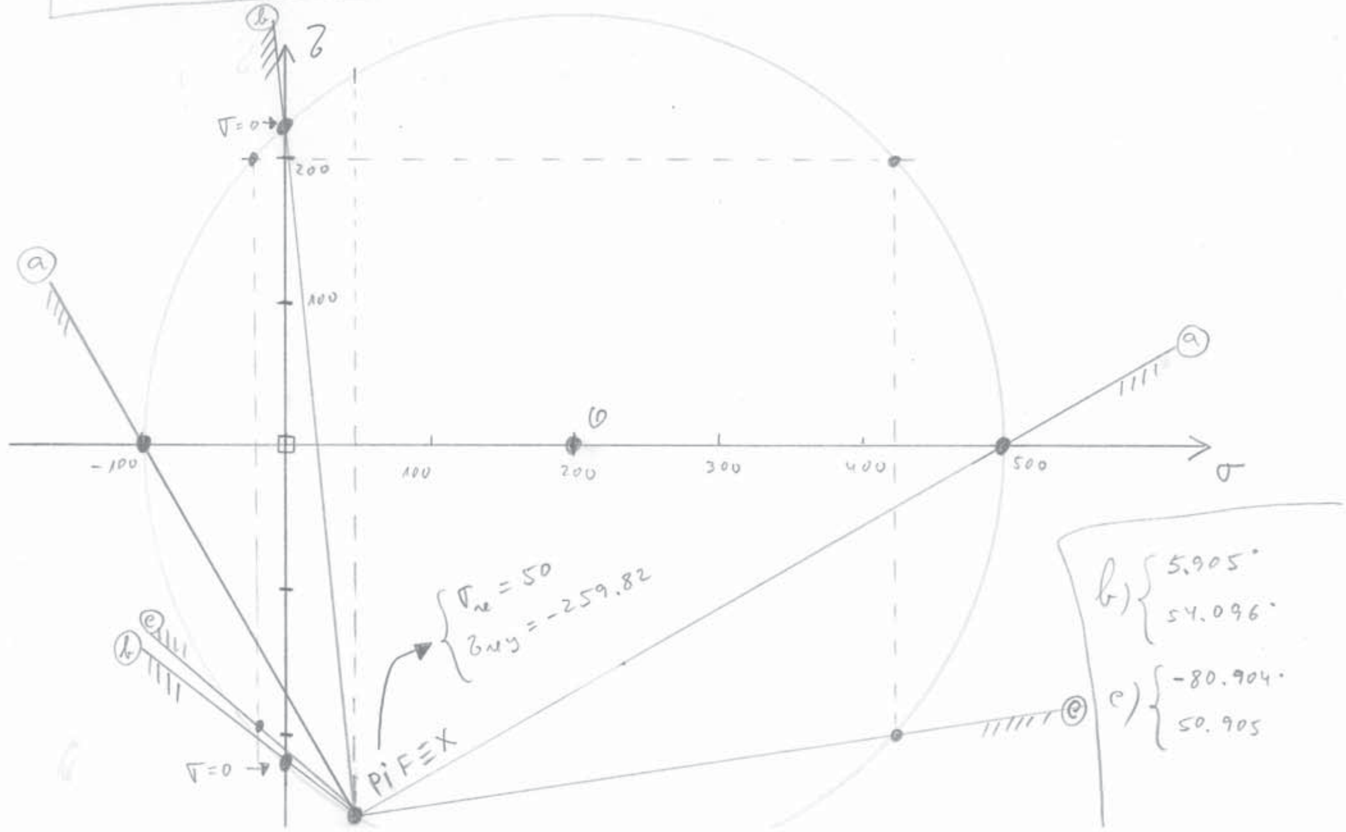
$$-100 = \sigma_2 = \quad " \quad " \quad " \quad " \qquad (\alpha = 30)$$

$$0 = \tau = \frac{\sigma_y - \sigma_x}{2}\,\mathrm{sen}(2\alpha) + \tau_{xy}\cos(2\alpha) \qquad (\alpha = -60)$$

0.5 + 0.5 cos(-120)	0.5 - 0.5 cos(-120)	sen(-120)	σ_x	=	500
0.5 + 0.5 cos(60)	0.5 - 0.5 cos(60)	sen(60)	σ_y		-100
-0.5 sen(-120)	0.5 sen(-120)	cos(-120)	τ_{xy}		0

0.25	0.75	-0.866025		500
0.75	0.25	0.866025		-100
0.433013	-0.433013	-0.5		0

$$\begin{cases} \sigma_x = 50 \\ \sigma_y = 350 \\ \tau_{xy} = -259.82 \end{cases}$$



b) $\begin{cases} 5.905° \\ 54.096° \end{cases}$

c) $\begin{cases} -80.904° \\ 50.905 \end{cases}$

R - 4. 2

(1) b) $\sigma = 0$

$$0 = \sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\alpha) + \tau_{xy} \operatorname{sen}(2\alpha)$$

$$0 = \frac{50 + 350}{2} + \frac{50 - 350}{2} \cos(2\alpha) - 259.82 \operatorname{sen}(2\alpha)$$

$$0 = 200 - 150 \cos(2\alpha) - 259.82 \operatorname{sen}(2\alpha) \qquad (1)$$

$$\begin{cases} 259.82\, s + 150\, c = 200 \\ s^2 + c^2 = 1 \end{cases} \qquad \begin{cases} s = \dfrac{200 - 150c}{259.82} \\ \left(\dfrac{200 - 150c}{259.82}\right)^2 + c^2 = 1 \end{cases}$$

$$(200 - 150c)^2 + 67\,506\, c^2 = 67\,506$$

$$40\,000 + 22500\, c^2 - 60\,000 c + 67\,506\, c^2 = 67\,506$$

$$90\,006\, c^2 - 60\,000\, c - 27\,506 = 0$$

$$c = \frac{60\,000 \pm \sqrt{60\,000^2 + 99\,028E5}}{180\,012} \quad \begin{cases} 0.97\,883 \\ -0.31221 \end{cases}$$

$$\cos(2\alpha) = \begin{cases} 0.97\,883 & \begin{cases} -5.905° \; (*) \\ \boxed{+5.905°} \end{cases} \\ -0.31221 & \begin{cases} -54.096° \; (*) \\ \boxed{+54.096°} \end{cases} \end{cases}$$

(*) Não é solução da equação (1)

R-4.3

(1) c)

$$200 = \tau = \frac{\sigma_y - \sigma_x}{2}\,\mathrm{sen}(2\alpha) + \tau_{xy}\cos(2\alpha)$$

$$200 = 150\,\mathrm{sen}(2\alpha) - 259.82\cos(2\alpha) \quad (1)$$

$$\begin{cases} 150\,s - 259.82\,c = 200 \\ s^2 + c^2 = 1 \end{cases} \qquad \begin{cases} s = \dfrac{200 + 259.82\,c}{150} \\ \left(\dfrac{200+259.82\,c}{150}\right)^2 + c^2 = 1 \end{cases}$$

$$(200 + 259.82c)^2 + 22500\,c^2 - 22500 = 0$$

$$40\,000 + 67\,506\,c^2 + 103\,928\,c + 22\,500\,c^2 - 22\,500 = 0$$

$$90\,006\,c^2 + 103\,928\,c + 17\,500 = 0$$

$$c = \frac{-103\,928 \pm \sqrt{103928^2 - 63\,004E5}}{180\,012} \begin{cases} -0.20466 \\ -0.95002 \end{cases}$$

$$\cos(2\alpha) = \begin{cases} -0.20466 \begin{cases} -50.905° \;(*) \\ +50.905° \end{cases} \\ -0.95002 \begin{cases} -80.904° \\ +80.904° \;(*) \end{cases} \end{cases}$$

(*) Não é solução da equação (1)

$$\alpha = 50.905° \Rightarrow \sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\sin(2\alpha)$$

$$\sigma = \frac{50+350}{2} + \frac{50-350}{2}\cos(2\times 50.905) - 259.82\sin(2\times 50.905)$$

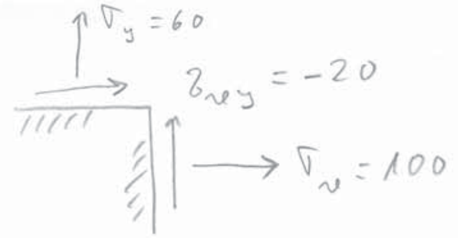
$$\sigma = -23.620$$

$$\alpha = -80.904° \Rightarrow \sigma = \frac{50+350}{2} + \frac{50-350}{2}\cos(-2\times 80.904) - 259.82\sin(-2\times 80.904)$$

$$\sigma = 423.619$$

R - 4.4

2

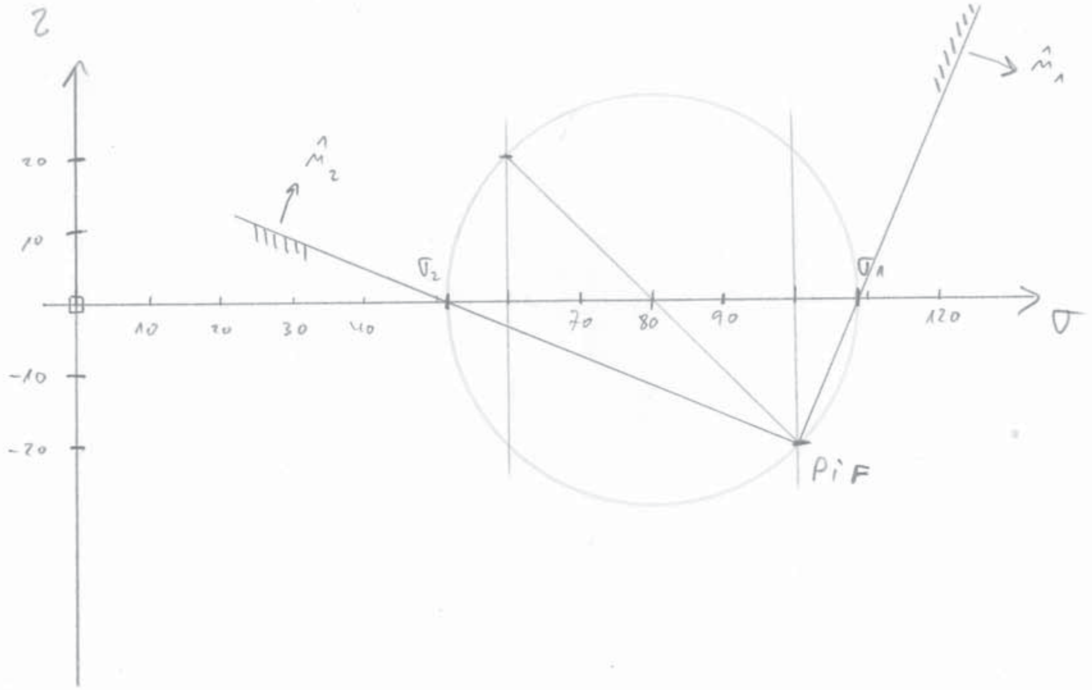


$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} =$$

$$= \frac{100+60}{2} \pm \sqrt{\left(\frac{100-60}{2}\right)^2 + (-20)^2} = \begin{cases} \sigma_1 = 108.28 \\ \sigma_2 = 51.72 \end{cases}$$

$$tg(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{-40}{100-60} = -1 \quad \begin{cases} \alpha = -22.5° \\ \alpha = 67.5° \end{cases}$$

$$\begin{cases} \alpha = -22.5 \Rightarrow \sigma = \frac{100+60}{2} + \frac{100-60}{2}\cos(2\alpha) - 20\sin(2\alpha) = 108.28 = \sigma_1 \\ \alpha = 67.5 \Rightarrow \sigma = \quad " \quad " \quad " \quad " \quad = 51.72 = \sigma_2 \end{cases}$$



R - 4.5

(3) a) $I_1 = \sigma_x + \sigma_y = 200$

$\alpha = 135° \Rightarrow \begin{cases} \sigma = 400 \\ \tau = 200 \end{cases}$

$$\begin{cases} 400 = \sigma = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos(2\alpha) + \tau_{xy}\,\mathrm{sen}(2\alpha) & (\alpha = 135°) \\ 200 = \tau = \dfrac{\sigma_y-\sigma_x}{2}\,\mathrm{sen}(2\alpha) + \tau_{xy}\cos(2\alpha) & (\alpha = 135°) \\ \sigma_x+\sigma_y = 200 \end{cases}$$

$$\begin{cases} 400 = 100 + \cancel{\dfrac{\sigma_x-\sigma_y}{2}\cos 270} + \tau_{xy}\,\mathrm{sen}\,270 \\ 200 = \dfrac{\sigma_y-\sigma_x}{2}\,\mathrm{sen}\,270 + \cancel{\tau_{xy}\cos 270} \\ \sigma_x+\sigma_y = 200 \end{cases} \quad \begin{cases} \tau_{xy} = -300 \\ \sigma_x - \sigma_y = 400 \\ \sigma_x+\sigma_y = 200 \end{cases} \quad \begin{cases} \tau_{xy} = -300 \\ \sigma_x = 300 \\ \sigma_y = -100 \end{cases}$$

$$\mathrm{tg}(2\alpha) = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} = \frac{-600}{300+100} = -1.5 \begin{cases} \alpha = -28.155° \\ \alpha = 61.845° \end{cases}$$

$$\alpha = -28.155 \Rightarrow \sigma = \frac{300-100}{2} + \frac{300+100}{2}\cos(2\alpha) - 300\,\mathrm{sen}(2\alpha) = 460.56 = \sigma_1$$

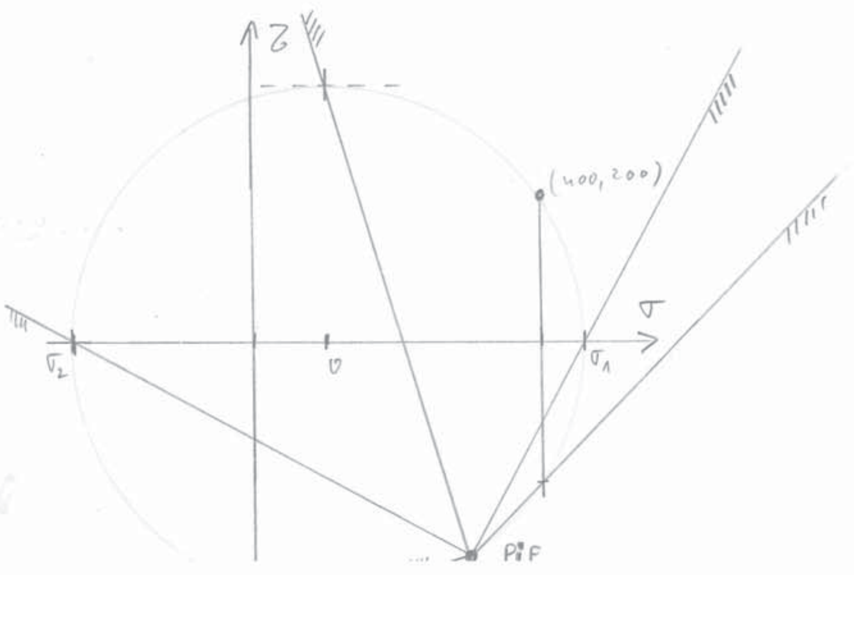
$$\alpha = 61.845 \Rightarrow \sigma = \quad " \quad " \quad " \quad " \quad = -260.56 = \sigma_2$$

---

b) $$\mathrm{tg}(2\alpha) = \frac{\sigma_y-\sigma_x}{2\tau_{xy}} = \frac{-100-300}{-600} = 0.66666 \begin{cases} \alpha = 16.845° \\ \alpha = -73.155 \end{cases}$$

$O = \dfrac{\sigma_x+\sigma_y}{2}$

$O = \dfrac{200}{2} = 100$



R-4.6

4 a) A $\begin{cases} \alpha = -30 \Rightarrow \sigma_1 = 6 \\ \alpha = 60 \Rightarrow \sigma_2 = -3 \end{cases}$ B $\begin{cases} \sigma_x = 4 \\ \sigma_y = -2 \\ \tau_{xy} = 2 \end{cases}$

Rodando A 30°:

$$\sigma_x = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos(2\times30) + 0\times \operatorname{sen}(2\times30) = 3.75$$

$$\sigma_y = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos(2\times120) + 0\times \operatorname{sen}(2\times120) = -0.75$$

$$\tau_{xy} = \frac{\sigma_2-\sigma_1}{2}\operatorname{sen}(2\times30) + 0\times\cos(2\times30) = -3.8971$$

Somando com B:

$$\left.\begin{aligned} \sigma_x &= 3.75+4 = 7.75 \\ \sigma_y &= -0.75-2 = -2.75 \\ \tau_{xy} &= -3.8971+2 = -1.8971 \end{aligned}\right\} \operatorname{tg}(2\alpha) = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} = -0.3614 \begin{cases} \alpha = -9.934° \\ \alpha = 80.066° \end{cases}$$

$$\alpha = -9.934 \Rightarrow \sigma = \frac{7.75-2.75}{2} + \frac{7.75+2.75}{2}\cos(2\alpha) - 1.8971\operatorname{sen}(2\alpha) = 8.082 = \sigma_1$$

$$\alpha = 80.066 \Rightarrow \sigma = \quad " \quad " \quad " \quad " = -3.082 = \sigma_2$$

---

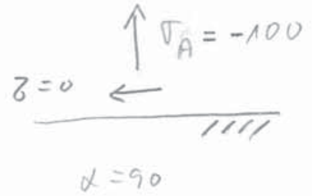
b)

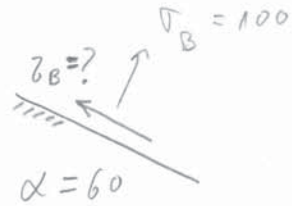
$$\alpha = -9.934+30 = 20.066 \Rightarrow \sigma = 2.5 + 5.25\cos(2\alpha) - 1.8971\operatorname{sen}(2\alpha) = 5.291$$

$$\Rightarrow \tau = \frac{-2.75-7.75}{2}\operatorname{sen}(2\alpha) - 1.8971\cos(2\alpha) = -4.834$$

R-4.7

5) a)





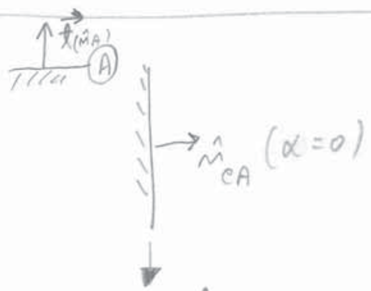
$$\begin{cases} \alpha = 90^\circ \Rightarrow -100 = \sigma = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\,\mathrm{sen}(2\alpha) \\ \alpha = 90^\circ \Rightarrow \quad 0 = \tau = \dfrac{\sigma_y - \sigma_x}{2}\,\mathrm{sen}(2\alpha) + \tau_{xy}\cos(2\alpha) \\ \alpha = 60 \Rightarrow 100 = \sigma = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\,\mathrm{sen}(2\alpha \end{cases}$$

$$\begin{cases} \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_y - \sigma_x}{2} = -100 \\ \tau_{xy} = 0 \\ \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}(-0.5) = 100 \end{cases} \quad \begin{cases} \sigma_y = -100 \\ \tau_{xy} = 0 \\ \dfrac{\sigma_x}{2} - 50 - \dfrac{\sigma_x}{4} - 25 = 100 \end{cases} \quad \begin{cases} \sigma_x = 700 \\ \sigma_y = -100 \\ \tau_{xy} = 0 \end{cases}$$

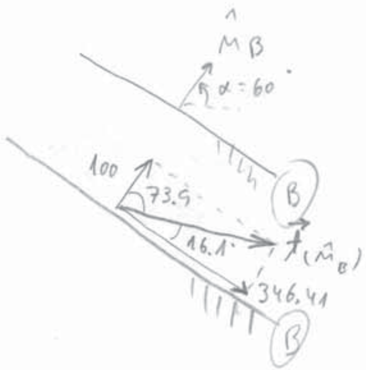
b)

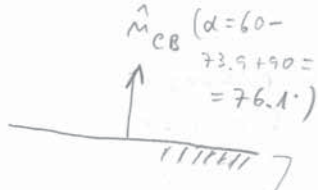
$$\alpha = 60^\circ \Rightarrow \tau_B = \frac{-100 - 700}{2}\,\mathrm{sen}\,120 + 0\cos 120 = -346.41$$

c)



Conjugada de A é // $\vec{t}_{(\hat{n}_A)}$

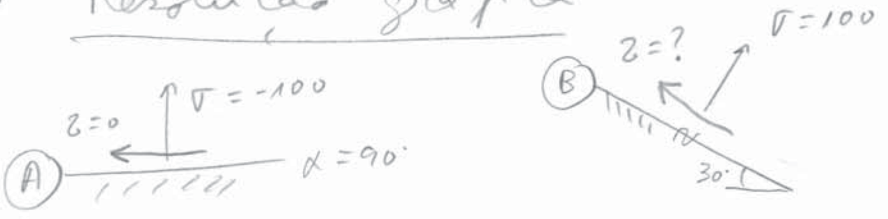


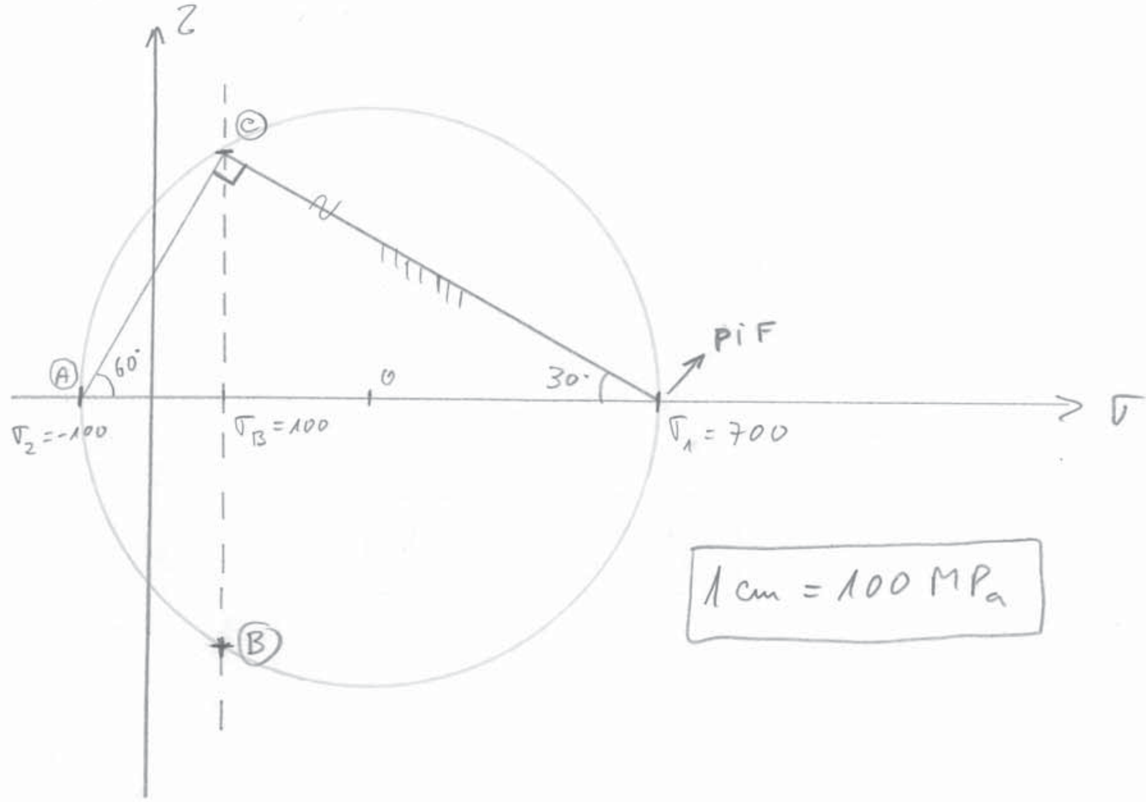


Conjugada de B é // $\vec{t}_{(\hat{n}_B)}$

R-4.8

(5) Resolução gráfica



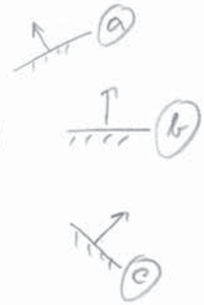


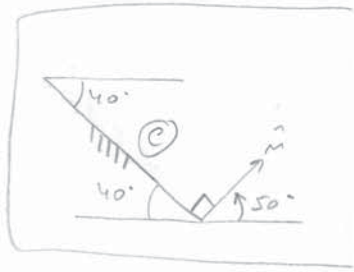
- (A) é uma faceta principal
- O polo irradiante está sobre o eixo "σ"
- Por (A) tira-se uma normal à faceta (B)
- Determina-se (C) (intersecção com a recta vertical $\sigma=100$)
- Tira-se uma normal a $\overline{AC}$ por (C) que intersecta o eixo σ no ponto $\sigma_1 = 700$
- O ponto (B) é o oposto de (C) ($\tau_B = -350$)

R-4.9

(6)

$$\begin{cases} \alpha = 120^\circ \Rightarrow \sigma = 100 \\ \alpha = 90 \Rightarrow \sigma = 200 \; (= \sigma_y) \\ \alpha = 50 \Rightarrow \sigma = 300 \end{cases}$$





$$\begin{cases} \sigma_x = ? \\ \sigma_y = 200 \\ \tau_{xy} = ? \end{cases}$$

$$\begin{cases} \sigma = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2} \cos(2\alpha) + \tau_{xy} \operatorname{sen}(2\alpha) \\ \tau = \dfrac{\sigma_y - \sigma_x}{2} \operatorname{sen}(2\alpha) + \tau_{xy} \cos(2\alpha) \end{cases}$$

$$\begin{cases} 100 = \dfrac{\sigma_x + 200}{2} + \dfrac{\sigma_x - 200}{2} \cos 240 + \tau_{xy} \operatorname{sen} 240 \\ 300 = \dfrac{\sigma_x + 200}{2} + \dfrac{\sigma_x - 200}{2} \cos 100 + \tau_{xy} \operatorname{sen} 100 \end{cases}$$

$$\begin{cases} \left(\frac{1}{2} + \frac{1}{2} \cos 240\right) \sigma_x + \operatorname{sen} 240 \; \tau_{xy} = 100 - 100 + 100 \cos 240 \\ \left(\frac{1}{2} + \frac{1}{2} \cos 100\right) \sigma_x + \operatorname{sen} 100 \; \tau_{xy} = 300 - 100 + 100 \cos 100 \end{cases}$$

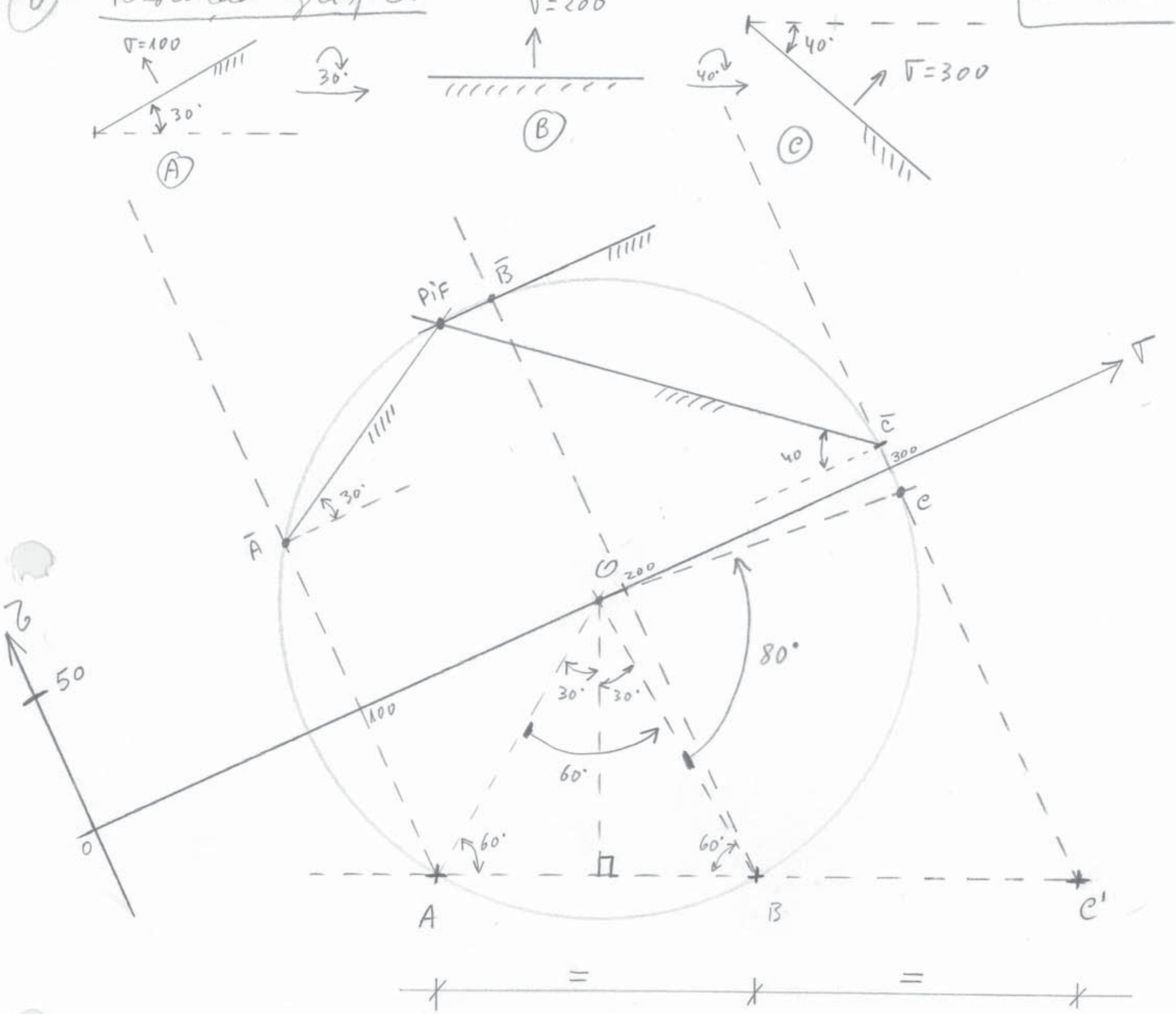
$$\begin{cases} 0.25 \; \sigma_x - 0.866\,025 \; \tau_{xy} = -50 \\ 0.413\,176 \; \sigma_x + 0.984\,808 \; \tau_{xy} = 182.635 \end{cases} \qquad \begin{cases} \sigma_x = 180.334 \\ \sigma_y = 200.000 \\ \tau_{xy} = 109.793 \end{cases}$$

$$\tau = \frac{200 - 180.334}{2} \operatorname{sen}(2\alpha) + 109.793 \cos(2\alpha)$$

$$\begin{cases} \alpha = 120^\circ \Rightarrow \tau = -63.412 \ \text{MPa} \\ \alpha = 90 \Rightarrow \tau = -109.793 \ \text{MPa} \\ \alpha = 50 \Rightarrow \tau = -9.382 \ \text{MPa} \end{cases}$$

⑥ Resolução gráfica

R – 4.10



→ Desenhar o triângulo (AOB) com uma escala qualquer, com os angulos indicados e com (AB) horizontal.

→ Desenhar o círculo de Mohr

→ Marcar o ponto C distante 80° de B

→ Marcar o ponto C' tal que $\|\vec{AB}\| = \|\vec{BC'}\|$

→ Traçar (C'C) ; ($B\bar{B}$) paralela a (C'C) ; ($A\bar{A}$) paralela a (C'C)

→ Traçar o eixo σ passando por O e perpendicularmente a (C'C)

→ Traçar o eixo τ sabendo que $\sigma_A = 100$, $\sigma_B = 200$ e $\sigma_C = 300$

→ Ligar $\bar{A}$, $\bar{B}$ e $\bar{C}$ ao PiF

→ Medir τ_A, τ_B e τ_C

MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

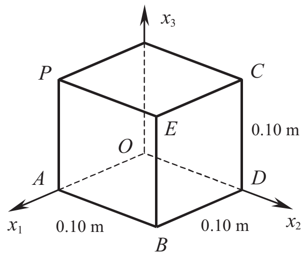
FOLHA 5 - ESTADO DE DEFORMAÇÃO

1 – Os pontos de um cubo de 10 cm de aresta apresentam deslocamentos dados por:

$$\vec{u} = \left(5 + 2x_1 - x_3, -7 - x_1, x_3\right) \times 10^{-2} (\text{cm})$$

a) Determine as novas coordenadas do ponto P.

b) Verifique se a transformação é afim e caracterize-a.



2 – O cubo representado na figura do problema 1 vai ser sujeito a uma transformação afim, caracterizada no referencial *S* pelos seguintes valores:

$$\vec{e}_0 = \vec{u}_0 = \left(0\ , 10^{-2}\ , 10^{-2}\right)\ (\text{metros})$$
$$e_{11} = e_{12} = e_{21} = 10^{-2}$$
$$e_{13} = e_{22} = 2\times 10^{-2}$$
$$e_{31} = e_{23} = 0$$
$$e_{33} = e_{32} = -2\times 10^{-2}$$

Determine:

a) As novas componentes do vector BC.

b) A nova distância entre B e C.

c) Os elementos do tensor das rotações e das deformações.

d) A direcção do eixo de rotação e o ângulo de rotação em graus.

e) Os novos comprimentos dos segmentos BD e DC.

f) Os novos valores dos ângulos BDC e ECD.

g) O volume do cubo após a transformação.

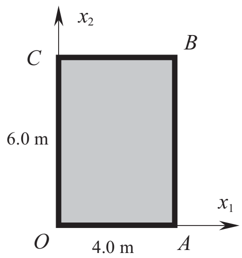
3 – Considere uma placa rectangular, que apresenta os deslocamentos indicados no quadro.

Calcule:

a) ε_1, ε_2, γ_{12}.

b) A rotação da placa em torno de O.

c) $(u_1)_B$; $(u_2)_B$.

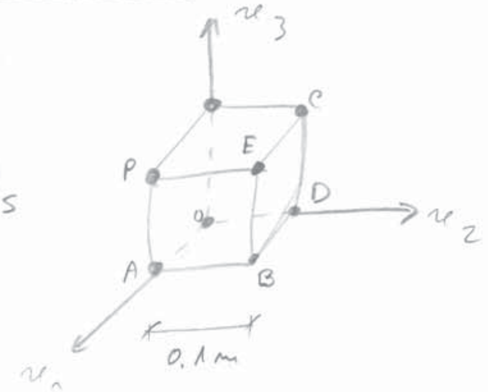


	x_1	x_2	
O	1	2	
A	2	6	$\times 10^{-5}\,m$
B	?	?	
C	2	4	

Resolução - FOLHA 5 - 1992/93

(1) a)

$$\vec{u} = (5 + 2x_1 - x_3,\ -7 - x_1,\ x_3)_S$$



$$\vec{OP}' = \vec{OP} + \vec{u}_P$$

$$\left.\begin{aligned} \vec{OP} &= (0.1,\ 0,\ 0.1) \\ \vec{u}_P &= (5.1,\ -7.1,\ 0.1) \end{aligned}\right\} \quad \vec{OP}' = (5.2,\ -7.1,\ 0.2)_S \ (m)$$

---

b) Tratar-se-á de uma transformação afim se for do tipo $u_i = e_{i0} + e_{ij} x_j$

com $\underbrace{e_{i0} = (u_i)_0 \quad ; \quad e_{ij} = \left(\frac{\partial u_i}{\partial x_j}\right)_0}_{\text{constantes}}$

$$\left.\begin{aligned} \vec{u}_0 &= (e_{10}, e_{20}, e_{30}) = (5, -7, 0) \\ \left[\frac{\partial u_i}{\partial x_j}\right] &= \begin{bmatrix} 2 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [e_{ij}] \end{aligned}\right\} \quad \underbrace{\vec{u} = \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{deformação homogénea}}$$

$$x_i' = x_i + u_i \Rightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \text{transformação afim}$$

R-5.2

(2)

a) $e_{i0} = \begin{bmatrix} 0 \\ 0.01 \\ 0.01 \end{bmatrix}$ $\quad e_{ij} = \begin{bmatrix} 0.01 & 0.01 & 0.02 \\ 0.01 & 0.02 & 0 \\ 0 & -0.02 & -0.02 \end{bmatrix}$

$x_i' = e_{i0} + (\delta_{ij} + e_{ij})\, x_j$ $\quad \begin{cases} B(0.1, 0.1, 0) \\ C(0, 0.1, 0.1) \end{cases}$

$$B' = \begin{bmatrix} 0 \\ 0.01 \\ 0.01 \end{bmatrix} + \begin{bmatrix} 1.01 & 0.01 & 0.02 \\ 0.01 & 1.02 & 0 \\ 0 & -0.02 & 0.98 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.102 \\ 0.113 \\ 0.008 \end{bmatrix}$$

$$C' = \begin{bmatrix} \\ '' \\ \\ \end{bmatrix} + \begin{bmatrix} \\ '' \\ \\ \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.003 \\ 0.112 \\ 0.106 \end{bmatrix}$$

$$\overrightarrow{B'C'} = \{x_{C'}\} - \{x_{B'}\} = \begin{bmatrix} -0.099 \\ -0.001 \\ 0.098 \end{bmatrix} \ (m)$$

---

(2) b) $\|\overrightarrow{B'C'}\| = \sqrt{0.099^2 + 0.001^2 + 0.098^2} = 0.1393\,0542\ m$

Resolução alternativa: $\varepsilon_{BC} = \hat{n}_{BC}^T \underset{\sim}{d}\, \hat{n}_{BC}$

$\vec{BC} = (0, 0.1, 0.1) - (0.1, 0.1, 0) = (-0.1, 0, 0.1) \Rightarrow \hat{n}_{BC} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$d_{ij} = \frac{1}{2}(e_{ij} + e_{ji}) = \frac{1}{2} \begin{bmatrix} 0.01 & 0.01 & 0.02 \\ 0.01 & 0.02 & 0 \\ 0 & -0.02 & -0.02 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0.01 & 0.01 & 0 \\ 0.01 & 0.02 & -0.02 \\ 0.02 & 0 & -0.02 \end{bmatrix} = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.02 & -0.01 \\ 0.01 & -0.01 & -0.02 \end{bmatrix}$$

$$\varepsilon_{BC} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.02 & -0.01 \\ 0.01 & -0.01 & -0.02 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{0.02}{\sqrt{2}} \\ -\frac{0.03}{\sqrt{2}} \end{bmatrix} = -0.015$$

$$\|\overrightarrow{B'C'}\| = \|\vec{BC}\| + \varepsilon_{BC}\|\vec{BC}\| = \sqrt{0.1^2 + 0.1^2} + (-0.015)\sqrt{0.1^2+0.1^2} = 0.1393\,0004\ m$$

2 c)

$$w_{ij} = \frac{1}{2}\left(e_{ij} - e_{ji}\right) = \frac{1}{2}\begin{bmatrix} 0.01 & 0.01 & 0.02 \\ 0.01 & 0.02 & 0 \\ 0 & -0.02 & -0.02 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0.01 & 0.01 & 0 \\ 0.01 & 0.02 & -0.02 \\ 0.02 & 0 & -0.02 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.01 \\ 0 & 0 & 0.01 \\ -0.01 & -0.01 & 0 \end{bmatrix}$$

$$d_{ij} = \frac{1}{2}\left(e_{ij} + e_{ji}\right) = \frac{1}{2}\begin{bmatrix} 0.01 & 0.01 & 0.02 \\ 0.01 & 0.02 & 0 \\ 0 & -0.02 & -0.02 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0.01 & 0.01 & 0 \\ 0.01 & 0.02 & -0.02 \\ 0.02 & 0 & -0.02 \end{bmatrix} = \begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.02 & -0.01 \\ 0.01 & -0.01 & -0.02 \end{bmatrix}$$

---

2 d) Vetor rotação: $\vec{w} = \left(\overset{w_{32}}{w_{23}}, \overset{w_{13}}{w_{31}}, \overset{w_{21}}{w_{12}}\right) = (0.01, -0.01, 0) \times (-1)$

Versor rotação: $\hat{n}_w = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \times (-1)$

Ângulo de rotação $\theta = \|\vec{w}\| = \sqrt{0.01^2 + 0.01^2} = 0.0141421$ rad

$$\theta = 0.0141421 \times \frac{180}{\pi} = 0.810285^\circ$$

---

2 e) $\vec{BD} \parallel e_1 \Rightarrow \varepsilon_{Bd} = \varepsilon_{11} = d_{11} = 0.01$

$\vec{DC} \parallel e_3 \Rightarrow \varepsilon_{DC} = \varepsilon_{33} = d_{33} = -0.02$

$\|\vec{B'D'}\| = \|\vec{BD}\|\left(1 + \varepsilon_{BD}\right) = 0.1\left(1 + 0.01\right) = 0.101$ m

$\|\vec{D'C'}\| = \|\vec{DC}\|\left(1 + \varepsilon_{DC}\right) = 0.1\left(1 - 0.02\right) = 0.098$ m

R-5.4

② f) $\measuredangle(BDC) \equiv \measuredangle(e_1 O e_3) \Rightarrow$ diminuição $= \gamma_{13}$

$$\gamma_{13} = 2 d_{13} = 2 \times 0.01 = 0.02 \text{ rad}$$

$$\measuredangle(BDC)' = \measuredangle(BDC) - \gamma_{13} = \frac{\pi}{2} - 0.02 = 1.550796 \text{ rad} = 88.8541°$$

$\measuredangle(ECD) \equiv \measuredangle(e_1 O -e_3) \Rightarrow$ diminuição $= -\gamma_{13} = -0.02$ rad

$$\measuredangle(ECD)' = \measuredangle(ECD) - (-0.02) = \frac{\pi}{2} + 0.02 = 1.590796 \text{ rad} = 91.1459°$$

---

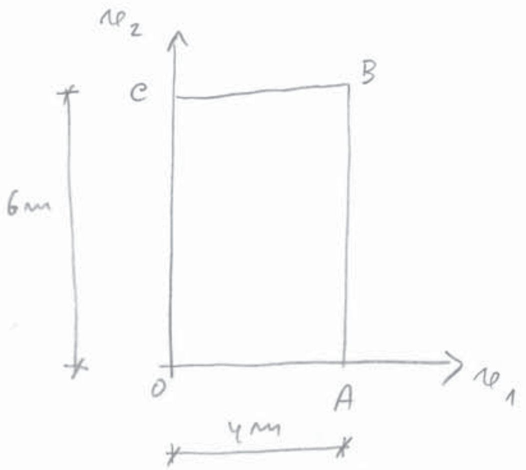
② g) $$\frac{\Delta V}{V} = I_1 = d_{11} + d_{22} + d_{33} = 0.01 + 0.02 - 0.02 = 0.01$$

$$V = 0.1^3 \Rightarrow \Delta V = 0.01 \times 0.1^3$$

$$V' = V + \Delta V = 0.1^3 + 0.01 \times 0.1^3 = 0.00101 \text{ m}^3$$

R - 5.5

3) a)



	u_1	u_2
O	1	2
A	2	6
B	?	?
C	2	4

$\times 10^{-5}$ (m)

Segmento $\overline{OA} \Rightarrow \boxed{\varepsilon_{11} = \frac{\Delta u_1}{\Delta x_1} = \frac{2-1}{4} \times 10^{-5} = \frac{1}{4} \times 10^{-5}}$

Segmento $\overline{OC} \Rightarrow \boxed{\varepsilon_{22} = \frac{\Delta u_2}{\Delta x_2} = \frac{4-2}{6} \times 10^{-5} = \frac{1}{3} \times 10^{-5}}$

Entre $\overline{OA}$ e $\overline{OC} \Rightarrow \boxed{\gamma_{12} = \frac{\Delta u_1^{(C)}}{\Delta x_2} + \frac{\Delta u_2^{(A)}}{\Delta x_1} = \left(\frac{2-1}{6} + \frac{6-2}{4}\right) \times 10^{-5} = \frac{7}{6} \times 10^{-5}}$

$d_{11} = \varepsilon_{11}$; $d_{22} = \varepsilon_{22}$; $d_{12} = d_{21} = \frac{\gamma_{12}}{2} = \frac{7}{12} \times 10^{-5}$

$$\underset{\sim}{d} = \begin{bmatrix} 1/4 & 7/12 \\ 7/12 & 1/3 \end{bmatrix} \times 10^{-5}$$

Alternativa: $u_i = e_{i0} + e_{ij} x_j$

Em A: $\begin{bmatrix} 2 \\ 6 \end{bmatrix} \times 10^{-5} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \times 10^{-5} + \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2 = 1 + 4 e_{11} \times 10^{5} \Rightarrow e_{11} = \frac{1}{4} \times 10^{-5} \\ 6 = 2 + 4 e_{21} \times 10^{5} \Rightarrow e_{21} = 1 \times 10^{-5} \end{cases}$

Em C: $\begin{bmatrix} 2 \\ 4 \end{bmatrix} \times 10^{-5} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \times 10^{-5} + \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} \Rightarrow \begin{cases} 2 = 1 + 6 e_{12} \times 10^{5} \Rightarrow e_{12} = \frac{1}{6} \times 10^{-5} \\ 4 = 2 + 6 e_{22} \times 10^{5} \Rightarrow e_{22} = \frac{1}{3} \times 10^{-5} \end{cases}$

$d_{ij} = \frac{1}{2}(e_{ij} + e_{ji}) = \frac{1}{2} \begin{bmatrix} 1/4 & 1/6 \\ 1 & 1/3 \end{bmatrix} \times 10^{-5} + \frac{1}{2} \begin{bmatrix} 1/4 & 1 \\ 1/6 & 1/3 \end{bmatrix} \times 10^{-5} = \begin{bmatrix} 1/4 & 7/12 \\ 7/12 & 1/3 \end{bmatrix} \times 10^{-5}$ ✓

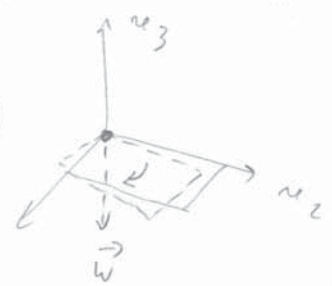
R - 5.6

③ b) Rotação em torno de O ?

$$e_{ij} = \begin{bmatrix} 1/4 & 1/6 \\ 1 & 1/3 \end{bmatrix} \times 10^{-5}$$

$$w_{ij} = \frac{1}{2}\left(e_{ij} - e_{ji}\right) = \frac{1}{2}\begin{bmatrix} 1/4 & 1/6 \\ 1 & 1/3 \end{bmatrix} \times 10^{-5} - \frac{1}{2}\begin{bmatrix} 1/4 & 1 \\ 1/6 & 1/3 \end{bmatrix} \times 10^{-5} = \begin{bmatrix} 0 & -5/12 \\ 5/12 & 0 \end{bmatrix} \times 10^{-5}$$

$$\vec{w} = (\overset{w_{32}}{w_{23}}, \overset{w_{13}}{w_{31}}, \overset{w_{21}}{w_{12}}) = \left(0, 0, -\frac{5}{12} \times 10^{-5}\right) \times (-1)$$



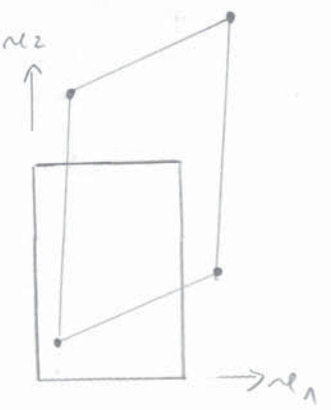
$$\|\vec{w}\| = \frac{5}{12} \times 10^{-5} = 4.166 \times 10^{-6} \text{ rad}$$

---

③ c) $u_{B_i} = e_{i0} + e_{ij} x_{B_j}$

$$\vec{u}_B = \begin{bmatrix} u_{B_1} \\ u_{B_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \times 10^{-5} + \begin{bmatrix} 1/4 & 1/6 \\ 1 & 1/3 \end{bmatrix} \times 10^{-5} \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\vec{u}_B = \begin{bmatrix} u_{B_1} \\ u_{B_2} \end{bmatrix} = \begin{bmatrix} 1 \times 10^{-5} + \left(\frac{1}{4} \times 4 + \frac{1}{6} \times 6\right) \times 10^{-5} \\ 2 \times 10^{-5} + \left(1 \times 4 + \frac{1}{3} \times 6\right) \times 10^{-5} \end{bmatrix} = \begin{bmatrix} 3 \times 10^{-5} \\ 8 \times 10^{-5} \end{bmatrix} \quad (m)$$



③ b) [FOLHA ⑤] Alternativa R-5.7

$$\vec{w} = (w_{32}, w_{13}, w_{21})$$

$$w_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = 0$$

$$w_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = 0$$

$$w_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \simeq \frac{1}{2}\left(\frac{\Delta u_2}{\Delta x_1} - \frac{\Delta u_1}{\Delta x_2}\right)$$

$$w_{21} = \frac{1}{2}\left(\frac{u_{A2} - u_{O2}}{\Delta x_1} - \frac{u_{C1} - u_{O1}}{\Delta x_2}\right) =$$

$$= \frac{1}{2}\left(\frac{6\times10^{-5} - 2\times10^{-5}}{4.0} - \frac{2\times10^{-5} - 1\times10^{-5}}{6.0}\right) =$$

$$= \frac{1}{2}\left(1\times10^{-5} - \frac{1}{6}\times10^{-5}\right) = \frac{5}{12}\times10^{-5}\ rad = w$$

---

③ c) $\underline{u} = \underline{u}^T + \underline{u}^R + u^D$

$$\underline{u}_B = \begin{bmatrix} 1\times10^{-5} \\ 2\times10^{-5} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{12}\times10^{-5} \\ \frac{5}{12}\times10^{-5} & 0 \end{bmatrix}\begin{bmatrix} 4.0 \\ 6.0 \end{bmatrix} + \begin{bmatrix} \frac{1}{4}\times10^{-5} & \frac{7}{12}\times10^{-5} \\ \frac{7}{12}\times10^{-5} & \frac{1}{3}\times10^{-5} \end{bmatrix}\begin{bmatrix} 4.0 \\ 6.0 \end{bmatrix} =$$

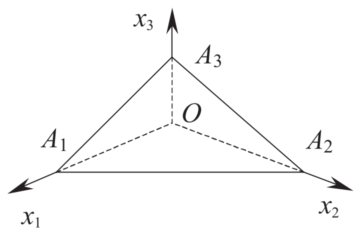
$$= \begin{bmatrix} 1\times10^{-5} \\ 2\times10^{-5} \end{bmatrix} + \begin{bmatrix} -\frac{5}{2}\times10^{-5} \\ \frac{5}{3}\times10^{-5} \end{bmatrix} + \begin{bmatrix} \frac{9}{2}\times10^{-5} \\ \frac{13}{3}\times10^{-5} \end{bmatrix} = \begin{bmatrix} 3\times10^{-5} \\ 8\times10^{-5} \end{bmatrix} \text{ (metros)}$$

MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

FOLHA 6 - ESTADO DE DEFORMAÇÃO

1 – Os pontos do tetraedro representado apresentam uma deformação caracterizada pelo tensor das deformações, cujos elementos são, no referencial $S \equiv (0, x_1, x_2, x_3)$

$$d_{11} = d_{22} = d_{33} = 10^{-2}$$
$$d_{12} = -2\times 10^{-2}$$
$$d_{13} = d_{23} = 0$$



$$\overline{OA_1} = \overline{OA_2} = 20\,cm$$
$$\overline{OA_3} = 10\,cm$$

a) Determine o novo comprimento de $\overline{A_1A_3}$.

b) Determine a área da face OA_1A_3 após a deformação.

c) Determine o novo ângulo $(A_1A_2A_3)$.

2 – Os deslocamentos dos pontos de um meio são dados por

$$\vec{u} = \left(2\alpha - \alpha x_1 x_3 , \beta - \alpha x_2 x_3 , \frac{1}{2}\alpha\left(x_1^2 + x_2^2\right) + \beta x_3^2 + \gamma\right)_S$$

em que α, β e γ são constantes.

Determine:

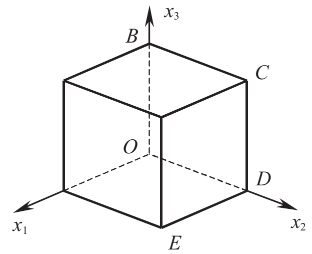
a) Os elementos do tensor das deformações em *S*.

b) O valor de α e β, sabendo que o segmento de recta elementar da vizinhança do ponto $P(1,1,1)$ inicialmente paralelo ao eixo x_3 apresenta uma extensão linear de 10^{-2} e uma rotação de $\sqrt{2}\times 10^{-2}$ rad.

3 – O paralelepípedo representado foi sujeito a uma deformação afim infinitesimal.

Sabendo que:

- a aresta BC mantém a direcção;
- a nova área da face O'B'C'D' é 15.3 cm^2;
- o vector $\overline{OE}$ depois de deformado tem as componentes $\overline{O'E'} = (4.04\,, 3.03\,, 0.01)_S$.



$\overline{BC} = 3\,cm$; $\overline{CD} = 5\,cm$; $\overline{DE} = 4\,cm$

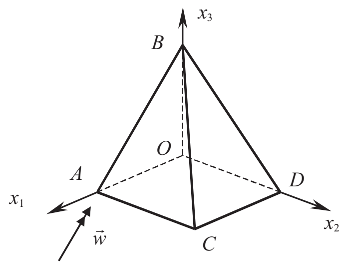
Determine:

a) Os elementos do tensor das deformações em S.

b) Os pares de segmentos que formam ângulos que se mantêm após a deformação.

4 – A pirâmide representada na figura está sujeita a um campo de deslocamentos linear resultante de:

- uma rotação de corpo rígido de $2\sqrt{2}\times 10^{-2}$ radianos, em torno de $\overline{AB}$ e no sentido indicado;
- uma translação de $5\sqrt{2}\times 10^{-2}$ cm ao longo da mesma recta e no sentido de A para B;
- um estado de deformação homogéneo, do qual se conhecem as componentes do tensor das deformações $d_{12} = 0$ e $d_{13} = 3\times 10^{-2}$ e o valor da extensão volumétrica $\varepsilon_v = 10^{-2}$.



$\overline{OA} = \overline{OB} = 3\,cm$; $\overline{OD} = 4\,cm$

Sabendo que após a deformação o comprimento de $\overline{OA}$ é de 3.03 cm, a área da base da pirâmide não se altera e que a distorção entre as direcções $\overline{AB}$ e $\overline{AC}$ é nula, determine:

a) As componentes em $(0, x_1, x_2, x_3)$ do tensor das deformações e do tensor das rotações.

b) O deslocamento do ponto C.

c) O comprimento de $\overline{BC}$ após a deformação.

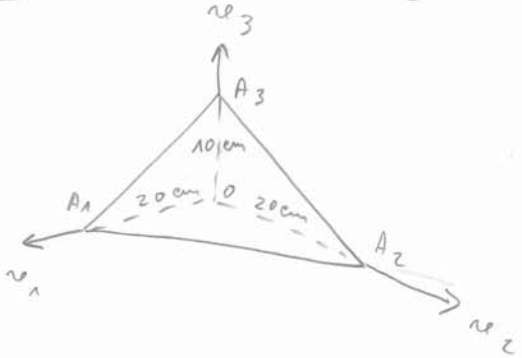
d) O valor da extensão principal máxima e a sua direcção de actuação.

Resolução - FOLHA 6 - 1992/93

R-6.1

(1) a)

$$\underset{\sim}{d} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times 10^{-2}$$



$$\overrightarrow{A_1A_3} = (0,0,10) - (20,0,0) = (-20,0,10) \Rightarrow \hat{n} = \frac{1}{\sqrt{5}}(-2,0,1)$$

$$\|\overrightarrow{A_1A_3}\| = \sqrt{20^2 + 10^2} = 10\sqrt{5} \text{ cm}$$

$$\varepsilon = \hat{n}^T \underset{\sim}{d}\, \hat{n} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0.01 & -0.02 & 0 \\ -0.02 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{0.02}{\sqrt{5}} \\ -\frac{0.04}{\sqrt{5}} \\ \frac{0.01}{\sqrt{5}} \end{bmatrix} = \frac{0.04}{5} + \frac{0.01}{5} = 0.01$$

$$\|\overrightarrow{A_1'A_3'}\| = \|\overrightarrow{A_1A_3}\|(1+\varepsilon) = 10\sqrt{5}(1+0.01) = 10.1\sqrt{5} \text{ cm}$$

---

(1) b)

$$\|\overrightarrow{OA_1'}\| = \|\overrightarrow{OA_1}\|(1+\varepsilon_{11}) = 20(1+0.01) = 20.2 \text{ cm}$$

$$\|\overrightarrow{OA_3'}\| = \|\overrightarrow{OA_3}\|(1+\varepsilon_{33}) = 10(1+0.01) = 10.1 \text{ cm}$$

$$\text{Área}(O A_1' A_3') = \frac{20.2 \times 10.1}{2} = 102.01 \text{ cm}^2$$

R-6.2

① c)
$$A_1' = \begin{bmatrix}20\\0\\0\end{bmatrix} + \begin{bmatrix}0.01 & -0.02 & 0\\ -0.02 & 0.01 & 0\\ 0 & 0 & 0.01\end{bmatrix}\begin{bmatrix}20\\0\\0\end{bmatrix} = (20.2, -0.4, 0)$$

$$A_2' = \begin{bmatrix}0\\20\\0\end{bmatrix} + \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}0\\20\\0\end{bmatrix} = (-0.4, 20.2, 0)$$

$$A_3' = \begin{bmatrix}0\\0\\10\end{bmatrix} + \begin{bmatrix} & '' & \end{bmatrix}\begin{bmatrix}0\\0\\10\end{bmatrix} = (0, 0, 10.1)$$

$$\vec{a} = A_1' - A_2' = (20.2, -0.4, 0) - (-0.4, 20.2, 0) = (20.6, -20.6, 0)$$

$$\vec{b} = A_3' - A_2' = (0, 0, 10.1) - (-0.4, 20.2, 0) = (0.4, -20.2, 10.1)$$

$$\measuredangle(A_1'A_2'A_3') = \arccos \frac{\vec{a} | \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} = \arccos\left(\frac{8.24 + 416.12 + 0}{\sqrt{20.6^2+20.6^2}\sqrt{0.4^2+20.2^2+10.1^2}}\right) = \underline{\underline{49.8435^\circ}}$$

Alternativa :
$$\begin{cases} A_1 - A_2 = (20,0,0) - (0,20,0) = (20,-20,0) \Rightarrow \hat{n}_a = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \\ A_3 - A_2 = (0,0,10) - (0,20,0) = (0,-20,10) \Rightarrow \hat{n}_b = \left(0, -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \end{cases}$$

$$\measuredangle(A_1A_2A_3) = \theta = \arccos(\hat{n}_a | \hat{n}_b) = \arccos\frac{2}{\sqrt{10}} = 50.7685^\circ$$

$$\varepsilon_a = \hat{n}_A^T \underset{\sim}{d}\, \hat{n}_A = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}0.01 & -0.02 & 0\\ -0.02 & 0.01 & 0\\ 0 & 0 & 0.01\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}\frac{0.03}{\sqrt{2}}\\ -\frac{0.03}{\sqrt{2}}\\ 0\end{bmatrix} = 0.03$$

$$\varepsilon_b = \hat{n}_B^T \underset{\sim}{d}\, \hat{n}_B = \begin{bmatrix}0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}0.01 & -0.02 & 0\\ -0.02 & 0.01 & 0\\ 0 & 0 & 0.01\end{bmatrix}\begin{bmatrix}0\\ -2/\sqrt{5}\\ 1/\sqrt{5}\end{bmatrix} = \begin{bmatrix}0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}\frac{0.04}{\sqrt{5}}\\ -\frac{0.02}{\sqrt{5}}\\ \frac{0.01}{\sqrt{5}}\end{bmatrix} = 0.01$$

$$K = \hat{n}_A^T \underset{\sim}{d}\, \hat{n}_B = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}0.01 & -0.02 & 0\\ -0.02 & 0.01 & 0\\ 0 & 0 & 0.01\end{bmatrix}\begin{bmatrix}0\\ -2/\sqrt{5}\\ 1/\sqrt{5}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}\frac{0.04}{\sqrt{5}}\\ -\frac{0.02}{\sqrt{5}}\\ \frac{0.01}{\sqrt{5}}\end{bmatrix} = \frac{0.06}{\sqrt{10}}$$

$$\gamma_{ab} = \frac{1}{\operatorname{sen}\theta}\left[2K - (\varepsilon_a + \varepsilon_b)\cos\theta\right] = 0.0163299 \text{ rad} = 0.935637^\circ$$

$$\measuredangle(A_1'A_2'A_3') = \measuredangle(A_1A_2A_3) - \gamma_{ab} = 50.7685^\circ - 0.9356^\circ = \underline{\underline{49.8329^\circ}}$$

R-6.3

(2)

a) $\vec{u} = \left(2\alpha - \alpha x_1 x_3 \ , \ \beta - \alpha x_2 x_3 \ , \ \frac{1}{2}\alpha\left(x_1^2 + x_2^2\right) + \beta x_3^2 + \gamma\right)$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{bmatrix} -\alpha x_3 & 0 & -\alpha x_1 \\ 0 & -\alpha x_3 & -\alpha x_2 \\ \alpha x_1 & \alpha x_2 & 2\beta x_3 \end{bmatrix}$$

$$d_{ij} = \frac{1}{2}\left(e_{ij} + e_{ji}\right) = \begin{bmatrix} -\alpha x_3 & 0 & 0 \\ 0 & -\alpha x_3 & 0 \\ 0 & 0 & 2\beta x_3 \end{bmatrix}$$

---

b)

$$w_{ij} = \frac{1}{2}\left(e_{ij} - e_{ji}\right) = \begin{bmatrix} 0 & 0 & -\alpha x_1 \\ 0 & 0 & -\alpha x_2 \\ \alpha x_1 & \alpha x_2 & 0 \end{bmatrix}$$

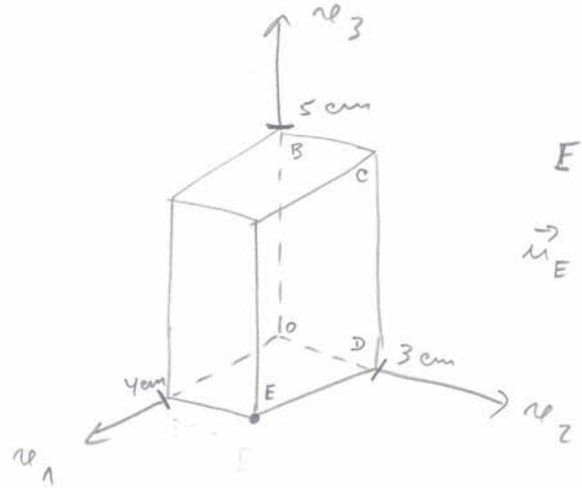
$\vec{w} = \left(w_{23}, w_{31}, w_{12}\right) = \left(-\alpha x_2, \alpha x_1, 0\right) \Rightarrow w = \|\vec{w}\| = \sqrt{\alpha^2 x_2^2 + \alpha^2 x_1^2} = \alpha\sqrt{x_1^2 + x_2^2}$

Como o referencial $Ox_1x_2x_3$ é principal, a rotação do segmento elementar // a x_3 e passando por $P(1,1,1)$ só pode ser de corpo rígido $\Rightarrow \sqrt{2} \times 10^{-2} = w = \alpha\sqrt{x_1^2 + x_2^2} = \alpha\sqrt{1^2 + 1^2} = \alpha\sqrt{2} \Rightarrow \alpha = 10^{-2}$

$10^{-2} = \varepsilon_{33} = 2\beta x_3 = 2\beta \times 1 = 2\beta \Rightarrow \beta = 0.5 \times 10^{-2}$

R-6.4

③ a)



$E(4,3,0)$ (cm)

$\vec{u}_E = (0.04, 0.03, 0.01)$ (cm)

$\overline{BC}$ mantém a direcção $\Rightarrow \hat{n}_2$ é direcção principal

$\overline{BC}$ mantém-se $\perp$ a $Ox_1x_3 \Rightarrow \gamma_{12} = \gamma_{23} = 0 \Rightarrow d_{12} = d_{23} = 0$

$$d_{ij} = \begin{bmatrix} d_{11} & 0 & d_{13} \\ 0 & d_{22} & 0 \\ d_{13} & 0 & d_{33} \end{bmatrix} \rightarrow 4 \text{ incógnitas}$$

$$u_i = d_{ij} x_j \Rightarrow \begin{bmatrix} 0.04 \\ 0.03 \\ 0.01 \end{bmatrix} = \begin{bmatrix} d_{11} & 0 & d_{13} \\ 0 & d_{22} & 0 \\ d_{13} & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} d_{11} = 0.01 \\ d_{22} = 0.01 \\ d_{13} = 0.0025 \end{cases}$$

$$S_1 + \Delta S_1 = (dx_2 + \varepsilon_2 dx_2)(dx_3 + \varepsilon_3 dx_3) = dx_2 dx_3 (1+\varepsilon_2)(1+\varepsilon_3)$$

$$S_1 + \Delta S_1 = S_1 (1 + \varepsilon_2 + \varepsilon_3 + \cancel{\varepsilon_2 \varepsilon_3}) = S_1 + S_1(\varepsilon_2 + \varepsilon_3)$$

$$\frac{\Delta S_1}{S_1} = \varepsilon_2 + \varepsilon_3$$

$$\frac{15.3 - 15}{15} = d_{22} + d_{33} = 0.01 + d_{33} \Rightarrow d_{33} = 0.01$$

$$d_{ij} = \begin{bmatrix} 0.01 & 0 & 0.0025 \\ 0 & 0.01 & 0 \\ 0.0025 & 0 & 0.01 \end{bmatrix}$$

R-6.5

③ b) Segmentos com ângulos que se mantêm $\Leftrightarrow$ direcções principais de deformação

$$\begin{cases} I_1 = 0.01 + 0.01 + 0.01 = 0.03 \\ I_2 = 0.01^2 + 0.01^2 + 0.01^2 - 0.0025^2 = 2.9375 \times 10^{-4} \\ I_3 = 0.01^3 - 0.01 \times 0.0025^2 = 9.375 \times 10^{-7} \end{cases}$$

$$\varepsilon^3 - I_1 \varepsilon^2 + I_2 \varepsilon - I_3 = 0 \Rightarrow \begin{cases} \varepsilon_I = 0.0125 \\ \varepsilon_{II} = 0.01 \\ \varepsilon_{III} = 0.0075 \end{cases}$$

$$\begin{bmatrix} 0.01-\varepsilon & 0 & 0.0025 \\ 0 & 0.01-\varepsilon & 0 \\ 0.0025 & 0 & 0.01-\varepsilon \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 0.01\, m_1 - \varepsilon m_1 + 0.0025\, m_3 = 0 \\ 0.01\, m_2 - \varepsilon m_2 = 0 \\ 0.0025\, m_1 + 0.01 m_3 - \varepsilon m_3 = 0 \\ m_1^2 + m_2^2 + m_3^2 = 1 \end{cases}$$

$$\varepsilon = 0.0075 \Rightarrow \begin{cases} 0.0025\, m_1 + 0.0025\, m_3 = 0 \Rightarrow m_3 = -m_1 \\ m_2 = 0 \\ m_3^2 + m_3^2 = 1 \end{cases} \quad \begin{Bmatrix} m_1 = -1/\sqrt{2} \\ m_2 = 0 \\ m_3 = 1/\sqrt{2} \end{Bmatrix} = \hat{m}_{III}$$

$$\varepsilon = 0.01 \Rightarrow \begin{Bmatrix} m_1 = 0 \\ m_2 = 1 \\ m_3 = 0 \end{Bmatrix} = \hat{m}_{II}$$

$$\hat{m}_I = \hat{m}_{II} \times \hat{m}_{III} = \begin{vmatrix} \cdot & \cdot & \cdot \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

(4) R - 6.6

$$\begin{cases} \varepsilon_x = 5 + x^2 + y^2 + x^4 + y^4 \rightarrow 2y + 4y^3 \rightarrow 2 + 12y^2 \\ \varepsilon_y = 6 + 3x^2 + 3y^2 + x^4 + y^4 \rightarrow 6x + 4x^3 \rightarrow 6 + 12x^2 \\ \gamma_{xy} = 10 + 4x^3 y + 4xy^3 + 8xy \rightarrow 12x^2 y + 4y^3 + 8y \rightarrow 12x^2 + 12y^2 + 8 \end{cases}$$

Verificação da equação de compatibilidade:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \Rightarrow 2 + 12y^2 + 6 + 12x^2 = 12x^2 + 12y^2 + 8$$

OK

Condições fronteira
$$\begin{cases} u(0,0) = 0 \\ v(0,0) = 0 \\ \text{rotação} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0 \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \text{ em } (0,0) \end{cases}$$

Integração:

$$\varepsilon_x = \frac{\partial u}{\partial x} \Rightarrow u(x,y) = \int \varepsilon_x \, dx + F_u(y)$$

$$u(x,y) = 5x + \frac{x^3}{3} + xy^2 + \frac{x^5}{5} + xy^4 + F_u(y)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = \int \varepsilon_y \, dy + F_v(x)$$

$$v(x,y) = 6y + 3x^2 y + y^3 + x^4 y + \frac{y^5}{5} + F_v(x)$$

→ $u(x,y)$ e $v(x,y)$ tem de respeitar a expressão de γ_{xy} e têm de respeitar as condições fronteira

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \begin{cases} \frac{\partial u}{\partial y} = 2xy + 4xy^3 + \frac{\partial F_u(y)}{\partial y} \\ \frac{\partial v}{\partial x} = 6xy + 4x^3 y + \frac{\partial F_v(x)}{\partial x} \end{cases}$$

R-6.7

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2xy + 4xy^3 + \frac{\partial F_u(y)}{\partial y} + 6xy + 4x^3y + \frac{\partial F_v(x)}{\partial x}$$

$$\gamma_{xy} = 8xy + 4xy^3 + 4x^3y + \frac{\partial F_u(y)}{\partial y} + \frac{\partial F_v(x)}{\partial x}$$

Dos dados: $\gamma_{xy} = 10 + 4x^3y + 4xy^3 + 8xy$

$$\underbrace{\frac{\partial F_u(y)}{\partial y}}_{\text{(função de } y)} + \underbrace{\frac{\partial F_v(x)}{\partial x}}_{\text{(função de } x)} = \underbrace{10}_{\text{(constante)}} \Rightarrow \text{soma de constantes} = \text{constante}$$

$$\left.\begin{aligned} F_u(y) &= ay + b \\ F_v(x) &= cx + d \end{aligned}\right\} \quad a + c = 10$$

$$\begin{cases} u(x,y) = 5x + \dfrac{x^3}{3} + xy^2 + \dfrac{x^5}{5} + xy^4 + ay + b \\ v(x,y) = 6y + 3x^2y + y^3 + x^4y + \dfrac{y^5}{5} + cx + d \end{cases}$$

Para $x = 0$; $y = 0$ vem $u = 0$; $v = 0$ $\Rightarrow b = 0$; $d = 0$

Para $x = 0$; $y = 0$ vem $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$

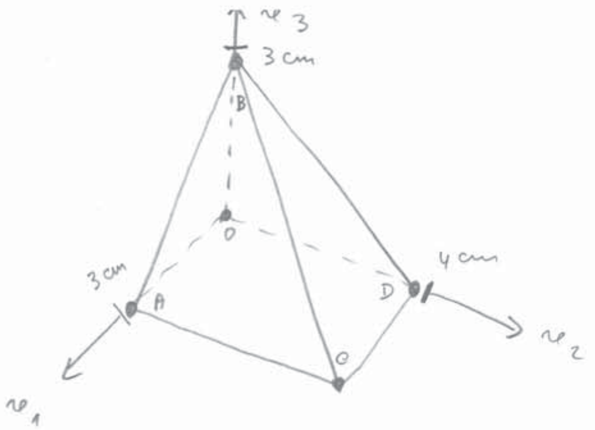
$$\frac{\partial u}{\partial y} = 2xy + 4xy^3 + a \; ; \; \frac{\partial v}{\partial x} = 6xy + 4x^3y + c \Rightarrow a = c$$

$$\begin{cases} a = c \\ a + c = 10 \end{cases} \quad \begin{cases} a = 5 \\ c = 5 \end{cases}$$

$$\begin{cases} u(x,y) = 5x + \dfrac{x^3}{3} + xy^2 + \dfrac{x^5}{5} + xy^4 + 5y \\ v(x,y) = 6y + 3x^2y + y^3 + x^4y + \dfrac{y^5}{5} + 5x \end{cases}$$

5 a)

R-6.8



$$d_{ij} = \begin{bmatrix} d_{11} & 0 & 0.03 \\ 0 & d_{22} & d_{23} \\ 0.03 & d_{23} & d_{33} \end{bmatrix} \qquad \varepsilon_v = d_{11} + d_{22} + d_{33} = 0.01$$

$$\Delta OA = OA\ \varepsilon_{e_1} = 3 d_{11} = 3.03 - 3 = 0.03 \Rightarrow d_{11} = 0.01$$

$$d_{11} + d_{22} + d_{33} = 0.01 \Rightarrow \cancel{0.01} + d_{22} + d_{33} = \cancel{0.01} \Rightarrow d_{33} = -d_{22}$$

$$d_{ij} = \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0 & d_{22} & d_{23} \\ 0.03 & d_{23} & -d_{22} \end{bmatrix} \rightarrow 2 \text{ incógnitas}$$

$$\frac{\Delta S_3}{S_3} = d_{11} + d_{22} = 0 \Rightarrow 0.01 + d_{22} = 0 \Rightarrow d_{22} = -0.01$$

$$\gamma_{\overline{AB}, \overline{AC}} = 0 \text{ e } \overline{AB} \perp \overline{AC} \Rightarrow \gamma_{\overline{AB}, \overline{AC}} = 2\, \hat{m}_{AB}^T\, \underset{\sim}{d}\, \hat{m}_{AC} = 0$$

$$2 \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0 & -0.01 & d_{23} \\ 0.03 & d_{23} & 0.01 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ -0.01 \\ d_{23} \end{bmatrix} = \frac{2 d_{23}}{\sqrt{2}} = 0 \Rightarrow d_{23} = 0$$

$$d_{ij} = \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0 & -0.01 & 0 \\ 0.03 & 0 & 0.01 \end{bmatrix}$$

R-6.9

$$\vec{w} = \|\vec{w}\| \left(+\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = 0.02\sqrt{2}\left(+\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$\vec{w} = (w_{23}, w_{31}, w_{12}) = (+0.02, 0, -0.02)$$

$$w_{ij} = \begin{bmatrix} 0 & -0.02 & 0 \\ +0.02 & 0 & +0.02 \\ 0 & -0.02 & 0 \end{bmatrix}$$

---

5 b) $\vec{u}_c = ?$ $\quad u_i = u_{i0} + w_{ij} x_j + d_{ij} x_j \qquad x_c = (3, 4, 0)$

$$\text{Translação} = 0.05\sqrt{2}\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = (-0.05, 0, 0.05) \text{ (cm)}$$

$$\vec{u}_c = \begin{bmatrix} -0.05 \\ 0 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 0 & -0.02 & 0 \\ 0.02 & 0 & 0.02 \\ 0 & -0.02 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0 & -0.01 & 0 \\ 0.03 & 0 & 0.01 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.10 \\ 0.02 \\ 0.06 \end{bmatrix} \text{ (cm)}$$

---

5 c) $\overline{BC} = C - B = (3, 4, 0) - (0, 0, 3) = (3, 4, -3) \rightarrow \|\vec{BC}\| = \sqrt{34}$

$$\hat{n}_{BC} = \frac{1}{\sqrt{34}}(3, 4, -3)$$

$$\varepsilon_{BC} = \hat{n}_{BC}^T \underset{\sim}{d}\, \hat{n}_{BC} = \begin{bmatrix} \frac{3}{\sqrt{34}} & \frac{4}{\sqrt{34}} & -\frac{3}{\sqrt{34}} \end{bmatrix} \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0 & -0.01 & 0 \\ 0.03 & 0 & 0.01 \end{bmatrix} \begin{bmatrix} 3/\sqrt{34} \\ 4/\sqrt{34} \\ -3/\sqrt{34} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{3}{\sqrt{34}} & \frac{4}{\sqrt{34}} & -\frac{3}{\sqrt{34}} \end{bmatrix} \begin{bmatrix} -0.06/\sqrt{34} \\ -0.04/\sqrt{34} \\ 0.06/\sqrt{34} \end{bmatrix} = -\frac{0.52}{34} = -\frac{0.26}{17}$$

$$\|\vec{B'C'}\| = \|\vec{BC}\| + \|\vec{BC}\|\, \varepsilon_{BC} = \sqrt{34} - \frac{0.26}{17}\sqrt{34} = 5.741773 \text{ cm}$$

5 d) Extensões principais R-6.10

$$d_{ij} = \begin{bmatrix} 0.01 & 0 & 0.03 \\ 0 & -0.01 & 0 \\ 0.03 & 0 & 0.01 \end{bmatrix}$$

$I_1 = 0.01 - 0.01 + 0.01 = 0.01$

$I_2 = -0.01^2 \underset{\sim}{-0.01^2} \underset{\sim}{+0.01^2} - 0.03^2 = -0.001$

$I_3 = -0.01^3 + 0.01 \times 0.03^2 = 8 \times 10^{-6}$

$$\varepsilon^3 - I_1 \varepsilon^2 + I_2 \varepsilon - I_3 = 0$$

$$\varepsilon^3 - 0.01\,\varepsilon^2 - 0.001\,\varepsilon - 8\times10^{-6} = 0 \Rightarrow \begin{cases} \varepsilon = 0.04 \text{ (máxima)} \\ \varepsilon = -0.01 \\ \varepsilon = -0.02 \end{cases}$$

$$\begin{bmatrix} 0.01-0.04 & 0 & 0.03 \\ 0 & -0.01-0.04 & 0 \\ 0.03 & 0 & 0.01-0.04 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -0.03\,n_1 + 0.03\,n_3 = 0 \\ -0.05\,n_2 = 0 \\ 0.03\,n_1 - 0.03\,n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1 \end{cases} \quad \begin{cases} n_1 = n_3 \\ n_2 = 0 \\ 2n_1^2 = 1 \end{cases} \quad \begin{cases} n_1 = \frac{1}{\sqrt{2}} \\ n_2 = 0 \\ n_3 = \frac{1}{\sqrt{2}} \end{cases}$$

$$\hat{n}_{\text{máxima}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

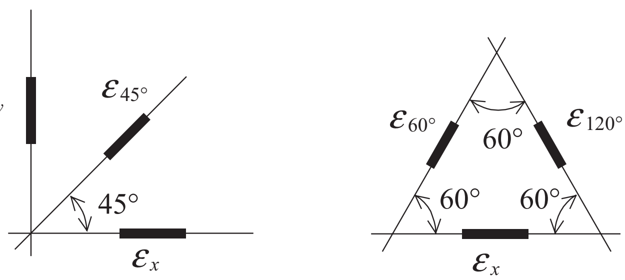
FOLHA 7 - ESTADO PLANO DE DEFORMAÇÃO

1 – Calcule as extensões principais e as direcções principais de um estado de deformação cujas componentes na origem são dadas por

$$\varepsilon_x = 5\times10^{-2} \;\; ; \;\; \varepsilon_y = 6\times10^{-2} \;\; ; \;\; \gamma_{xy} = 10\times10^{-2}.$$

Usando a construção de Mohr, verifique os resultados obtidos.

2 – Considere as duas seguintes montagens habituais em extensometria

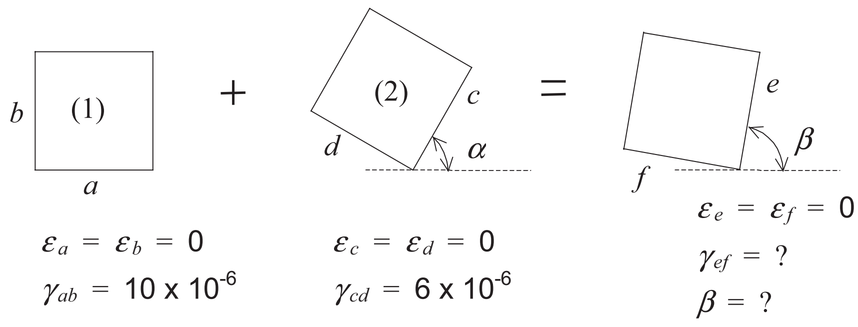


(Roseta em estrela) (Roseta equiangular)

Obtenha as extensões principais e as direcções principais de deformação que lhes estão associadas para as seguintes leituras:

- Roseta em estrela: $\varepsilon_x = -1.5\times10^{-3}$; $\varepsilon_y = 10^{-3}$; $\varepsilon_{45°} = -2\times10^{-3}$
- Roseta equiangular: $\varepsilon_x = 270\times10^{-6}$; $\varepsilon_{60°} = 370\times10^{-6}$; $\varepsilon_{120°} = -120\times10^{-6}$

3 – São definidos dois estados de distorção, representados na figura.



Demonstre que a soma destes dois estados de deformação é ainda um estado de deformação distorcional e calcule a correspondente distorção.

4 – Considere-se o estado plano de deformação traduzido pelas equações

$$\varepsilon_x = 5 + x^2 + y^2 + x^4 + y^4$$
$$\varepsilon_y = 6 + 3x^2 + 3y^2 + x^4 + y^4$$
$$\gamma_{xy} = 10 + 4x^3 y + 4xy^3 + 8xy$$

Supondo que as condições de apoio são tais que impedem as translações na origem, bem como a rotação, determine os campos de deslocamentos u e v.

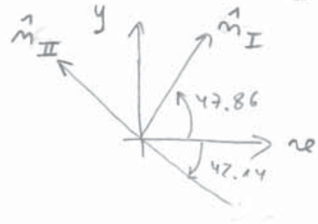
Resolução - FOLHA 7 - 1992/93

(1) $\varepsilon_x = 5 \; ; \; \varepsilon_y = 6 \; ; \; \gamma_{xy} = 10 \Rightarrow \frac{\gamma_{xy}}{2} = 5$

$$\begin{cases} \varepsilon = \varepsilon_x \cos^2\alpha + \varepsilon_y \operatorname{sen}^2\alpha + \frac{\gamma_{xy}}{2} \operatorname{sen} 2\alpha \\ \frac{\gamma}{2} = \frac{\varepsilon_y - \varepsilon_x}{2} \operatorname{sen} 2\alpha + \frac{\gamma_{xy}}{2} \cos 2\alpha \end{cases}$$

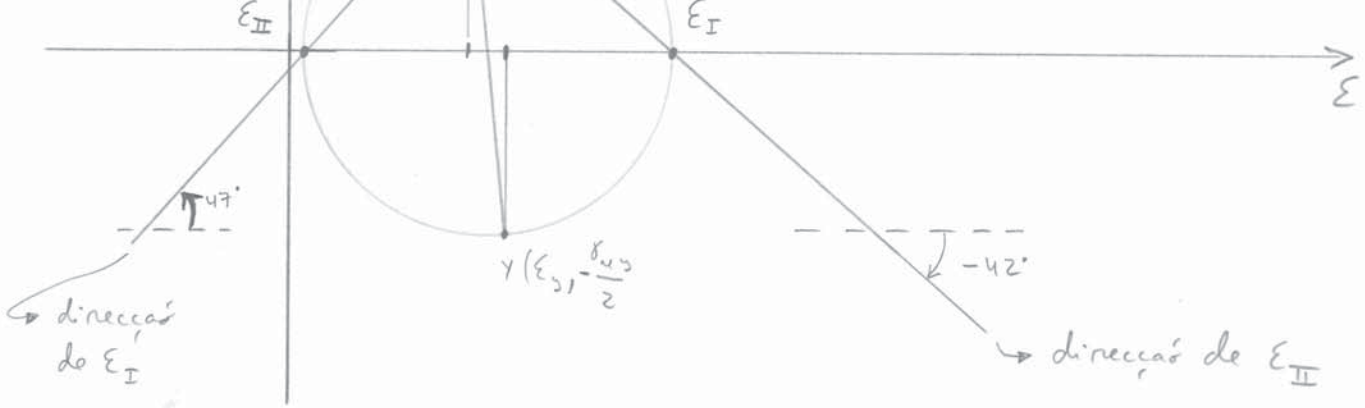
$$\frac{\gamma}{2} = 0 \Rightarrow (\varepsilon_x - \varepsilon_y) \operatorname{sen} 2\alpha = \gamma_{xy} \cos 2\alpha \Rightarrow \operatorname{tg} 2\alpha = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\operatorname{tg} 2\alpha = \frac{10}{5-6} \Rightarrow \alpha = \begin{cases} -42.14° \\ 47.86° \end{cases}$$



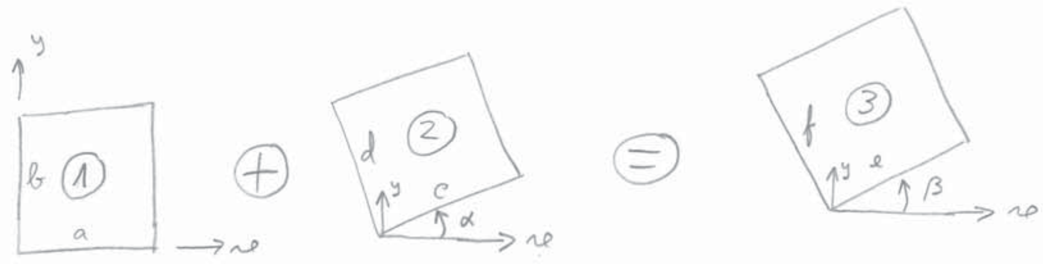
$$\alpha = -42.14 \Rightarrow \varepsilon = 5\cos^2\alpha + 6\operatorname{sen}^2\alpha + 5\operatorname{sen}2\alpha = 0.4751 = \varepsilon_{II}$$

$$\alpha = 47.86 \Rightarrow \varepsilon = 5\cos^2\alpha + 6\operatorname{sen}^2\alpha + 5\operatorname{sen}2\alpha = 10.5249 = \varepsilon_I$$



(2)

R-7.2



$\varepsilon_a = 0$ $\quad$ $\varepsilon_c = 0$ $\quad$ $\varepsilon_e = 0$

$\varepsilon_b = 0$ $\quad$ $\varepsilon_d = 0$ $\quad$ $\varepsilon_f = 0$

γ_{ab} qualquer $\quad$ γ_{cd} qualquer $\quad$ $\gamma_{ef} = ?$

$\beta = ?$

Rodar (2) $-\alpha$

$$\begin{cases} \varepsilon_x^{(2)} = \overset{=0}{\varepsilon_c} \cos^2(-\alpha) + \overset{=0}{\varepsilon_d} \operatorname{sen}^2(-\alpha) + \dfrac{\gamma_{cd}}{2} \operatorname{sen}(-2\alpha) = -\dfrac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha \\ \varepsilon_y^{(2)} = \overset{=0}{\varepsilon_c} \cos^2(90-\alpha) + \overset{=0}{\varepsilon_d} \operatorname{sen}^2(90-\alpha) + \dfrac{\gamma_{cd}}{2} \operatorname{sen} 2(90-\alpha) = \dfrac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha \\ \dfrac{\gamma_{xy}^{(2)}}{2} = \dfrac{\overset{=0}{\varepsilon_d} - \overset{=0}{\varepsilon_c}}{2} \operatorname{sen}(-2\alpha) + \dfrac{\gamma_{cd}}{2} \cos(-2\alpha) = \dfrac{\gamma_{cd}}{2} \cos 2\alpha \end{cases}$$

Somar (1) com (2) rodado:

$$\begin{cases} \left.\begin{array}{l} \varepsilon_x^{(3)} = -\dfrac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha = d_{11} \\ \varepsilon_y^{(3)} = \dfrac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha = d_{22} \end{array}\right\} \Rightarrow \dfrac{\Delta V}{V} = d_{11} + d_{22} = 0 \Rightarrow \text{estado de deformação distorcional} \\ \dfrac{\gamma_{xy}^{(3)}}{2} = \dfrac{\gamma_{ab}}{2} + \dfrac{\gamma_{cd}}{2} \cos 2\alpha \end{cases}$$

R - 7.3

Cálculo de β

Procurar o ângulo β que anula ε_e

Como $\varepsilon_e + \varepsilon_f = d_{11} + d_{22} = 0$, $\varepsilon_e = 0 \Rightarrow \varepsilon_f = 0$

$$\varepsilon_e = \varepsilon_x^{③} \cos^2\beta + \varepsilon_y^{③} \operatorname{sen}^2\beta + \frac{\gamma_{xy}^{③}}{2} \operatorname{sen} 2\beta = 0$$

$$-\frac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha \cos^2\beta + \frac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha \operatorname{sen}^2\beta + \left(\frac{\gamma_{ab}}{2} + \frac{\gamma_{cd}}{2}\cos 2\alpha\right) \operatorname{sen} 2\beta = 0$$

$$\operatorname{sen} 2\beta \left(\frac{\gamma_{ab}}{2} + \frac{\gamma_{cd}}{2} \cos 2\alpha\right) = \frac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha \left(\cos^2\beta - \operatorname{sen}^2\beta\right)$$

$$\operatorname{tg} 2\beta = \frac{\frac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha}{\frac{\gamma_{ab}}{2} + \frac{\gamma_{cd}}{2} \cos 2\alpha}$$

$$\boxed{\beta = \frac{1}{2} \operatorname{arctg}\left(\frac{\gamma_{cd} \operatorname{sen} 2\alpha}{\gamma_{ab} + \gamma_{cd} \cos 2\alpha}\right)}$$

$$\frac{\gamma_{ef}}{2} = \frac{\varepsilon_y^{③} - \varepsilon_x^{③}}{2} \operatorname{sen} 2\beta + \frac{\gamma_{xy}^{③}}{2} \cos 2\beta =$$

$$= \frac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha \operatorname{sen} 2\beta + \left(\frac{\gamma_{ab}}{2} + \frac{\gamma_{cd}}{2} \cos 2\alpha\right) \cos 2\beta =$$

$$= \frac{\gamma_{cd}}{2} \operatorname{sen} 2\alpha \operatorname{sen} 2\beta + \frac{\gamma_{ab}}{2} \cos 2\beta + \frac{\gamma_{cd}}{2} \cos 2\alpha \cos 2\beta =$$

$$\boxed{\frac{\gamma_{ef}}{2} = \frac{\gamma_{ab}}{2} \cos 2\beta + \frac{\gamma_{cd}}{2} \cos\left[2(\alpha - \beta)\right]}$$

R-7.4

(3) a)
$$\begin{cases} \varepsilon_x = -1.5 \times 10^{-3} \\ \varepsilon_y = 10^{-3} \\ \varepsilon_{45} = -2 \times 10^{-3} \end{cases}$$

$$\boxed{\varepsilon = \varepsilon_x \cos^2\alpha + \varepsilon_y \operatorname{sen}^2\alpha + \frac{\gamma_{xy}}{2} \operatorname{sen} 2\alpha}$$

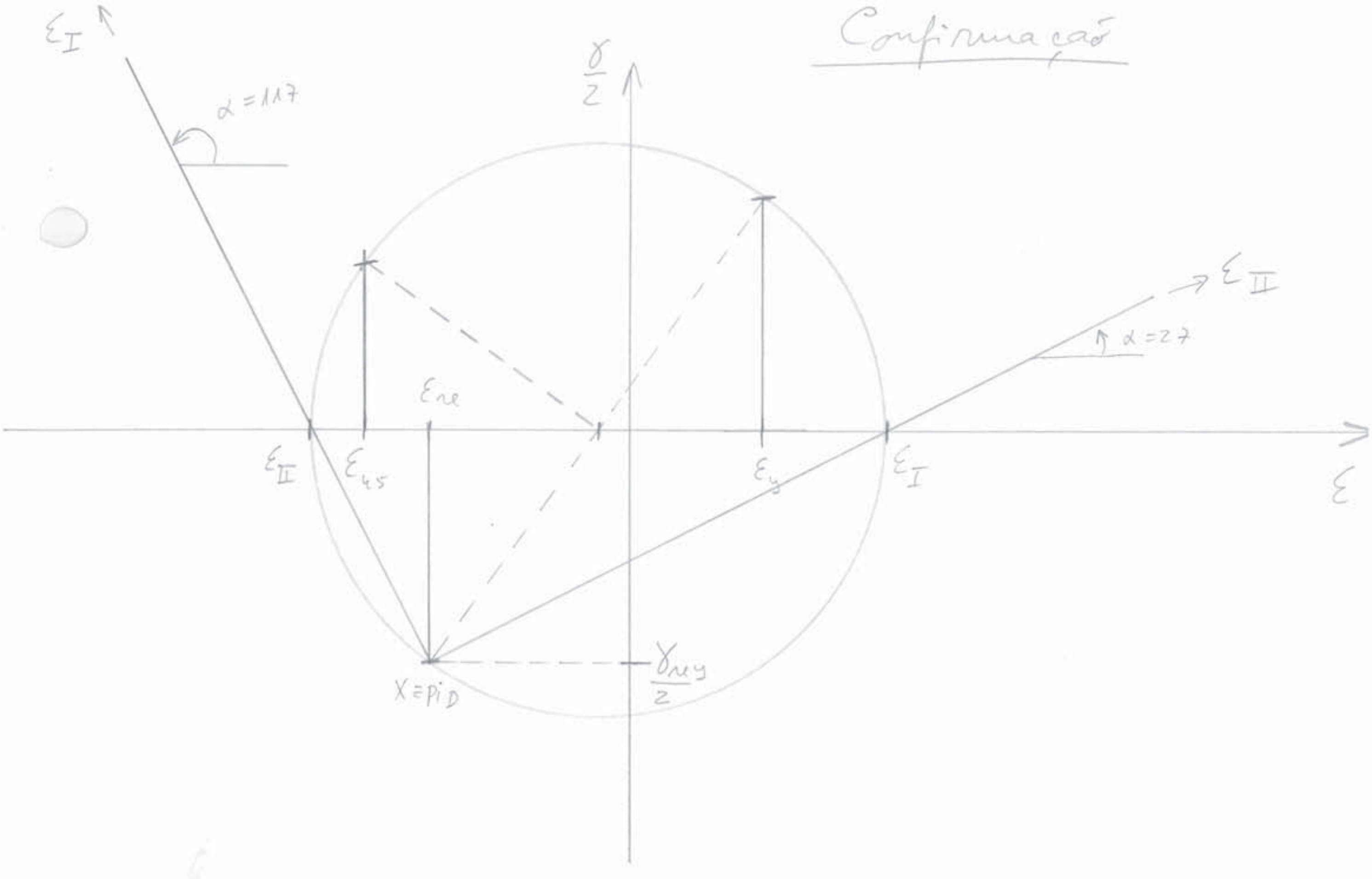
$$\alpha = 45° \Rightarrow \varepsilon_{45} = -2 \times 10^{-3} = -1.5 \times 10^{-3}\, \frac{1}{2} + 10^{-3}\, \frac{1}{2} + \frac{\gamma_{xy}}{2} \Rightarrow \gamma_{xy} = -3.5 \times 10^{-3}$$

$$\frac{\gamma_{xy}}{2} = -1.75 \times 10^{-3}$$

$$\operatorname{tg} 2\alpha = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{-3.5 \times 10^{-3}}{-1.5 \times 10^{-3} - 10^{-3}} = 1.4 \Rightarrow \alpha = \begin{cases} 27.231° \\ 117.231° \end{cases}$$

$$\begin{cases} \alpha = 27.231 \Rightarrow \varepsilon = -0.002401 = \varepsilon_{II} \\ \alpha = 117.231 \Rightarrow \varepsilon = 0.001901 = \varepsilon_{I} \end{cases}$$

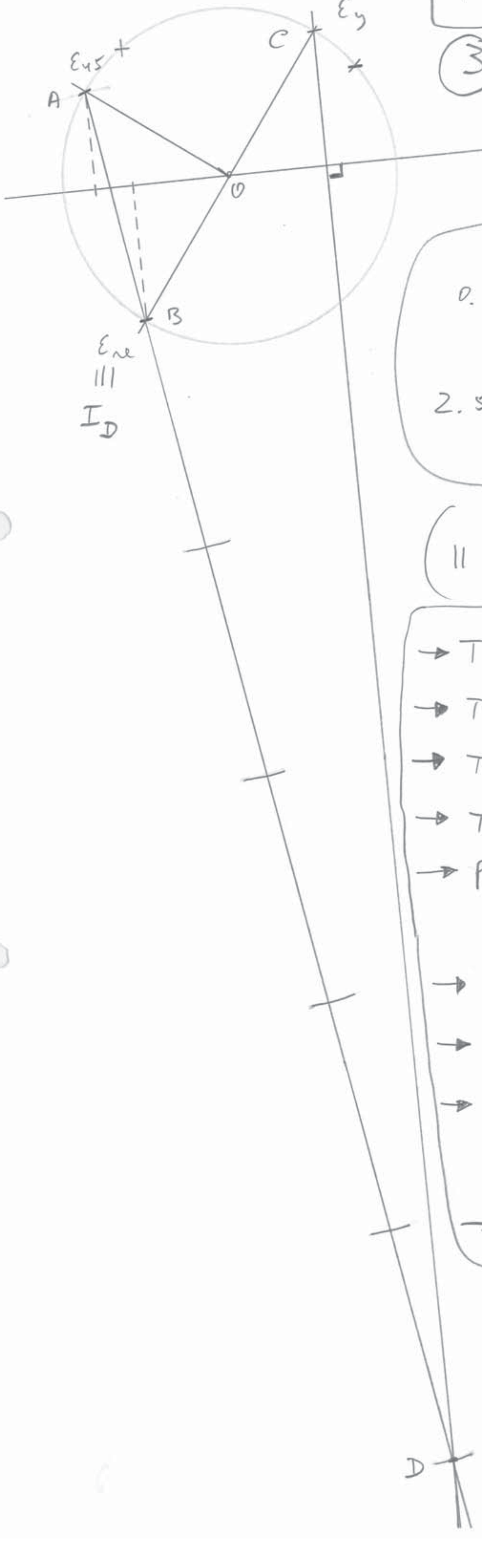
Confirmação



FOLHA 7

3 a)

Resolução gráfica



$$0.5 \times 10^{-3} \quad \varepsilon_{45} = -2 \times 10^{-3}$$

$$\varepsilon_{xx} = -1.5 \times 10^{-3}$$

$$2.5 \times 10^{-3} \quad \varepsilon_y = 1 \times 10^{-3}$$

$$\left(\| BD \| = 5 \times \| AB \| \right) \circledast$$

- → Traçar uma circunferência qualquer.
- → Traçar o diâmetro $\overline{BC}$ qualquer.
- → Traçar $\overline{OA}$ normal a $\overline{BC}$.
- → Traçar $\overline{AB}$.
- → Prolongar $\overline{AB}$ até ao ponto D respeitando $\circledast$.
- → Traçar $\overline{CD}$.
- → Traçar o eixo ε normal a $\overline{CD}$.
- → Traçar a linha de chamada do ponto A normal ao eixo ε.
- → Idem para o ponto B.

R-7.5

(3) b)

$$\begin{cases} \varepsilon_x = 270 \times 10^{-6} \\ \varepsilon_{60} = 370 \times 10^{-6} \\ \varepsilon_{120} = -120 \times 10^{-6} \end{cases}$$

$$\boxed{\varepsilon = \varepsilon_x \cos^2\alpha + \varepsilon_y \operatorname{sen}^2\alpha + \frac{\gamma_{xy}}{2} \operatorname{sen} 2\alpha}$$

$$\alpha = 60 \Rightarrow \varepsilon_{60} = 370 = 270 \cos^2 60 + \varepsilon_y \operatorname{sen}^2 60 + \frac{\gamma_{xy}}{2} \operatorname{sen} 120$$

$$\alpha = 120 \Rightarrow \varepsilon_{120} = -120 = 270 \cos^2 120 + \varepsilon_y \operatorname{sen}^2 120 + \frac{\gamma_{xy}}{2} \operatorname{sen} 240$$

$$\begin{cases} 0.75\,\varepsilon_y + 0.433013\,\gamma_{xy} = 302.5 \\ 0.75\,\varepsilon_y - 0.433013\,\gamma_{xy} = -187.5 \end{cases} \quad \begin{cases} \varepsilon_y = 76.6666 \times 10^{-6} \\ \gamma_{xy} = 565.803 \times 10^{-6} \end{cases}$$

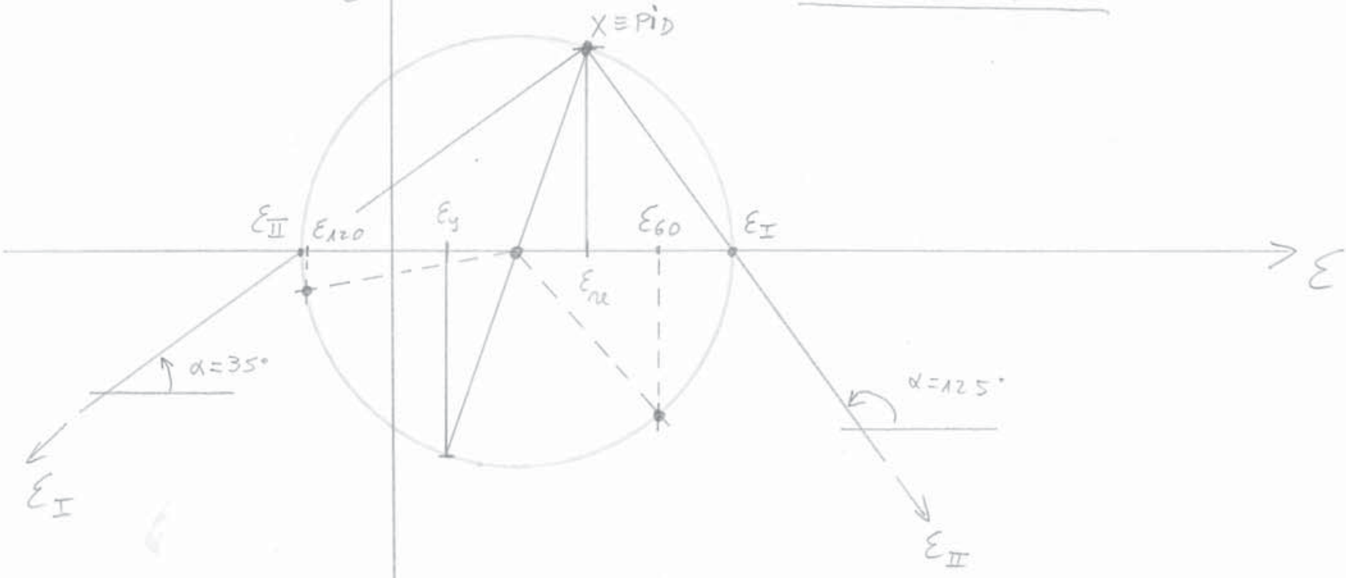
$$\left(\frac{\gamma_{xy}}{2} = 282.902 \times 10^{-6}\right)$$

$$\operatorname{tg} 2\alpha = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{565.803}{270 - 76.6666} = 2.926567 \Rightarrow \alpha = \begin{cases} 35.567^\circ \\ 125.567^\circ \end{cases}$$

$$\alpha = 35.567^\circ \Rightarrow \varepsilon = 472.294 \times 10^{-6} = \varepsilon_I$$

$$\alpha = 125.567 \Rightarrow \varepsilon = -125.628 \times 10^{-6} = \varepsilon_{II}$$

Confirmação



Alternativa

FOLHA 8

2

$$\begin{Bmatrix} d_{11} = -0.01 \\ d_{22} = -0.01 \\ d_{33} = ? \end{Bmatrix} \quad I_1^d = -0.02 + d_{33}$$

$$\begin{cases} E = 200\,000 \text{ MPa} \\ \nu = 0.25 \end{cases}$$

$$I_1^d = \frac{1-2\nu}{E} I_1^{\sigma}$$

$$-0.02 + d_{33} = \frac{1-0.5}{200\,000}(-4000)$$

$$d_{33} = \frac{1}{400\,000}(-4000) + 0.02 = -0.01 + 0.02 = 0.01$$

Em seguida calcular as extensões principais

MECÂNICA DOS SÓLIDOS - ANO LECTIVO 2002/2003 - 2.ANO - 1.SEM.

FOLHA 8 - RELAÇÕES ENTRE TENSÕES E DEFORMAÇÕES

1 – Um corpo perfeitamente elástico, homogéneo e isotrópico encontra-se sujeito a um estado de tensão cujas componentes são dadas no referencial S por

$$[\tau] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times 10^2 \text{ MPa} .$$

Os parâmetros elásticos do material são: $\nu = \dfrac{1}{4}$; $E = 10^5$ MPa .

Que pressão deve ser aplicada ao corpo (estado de tensão isotrópico) para que ele recupere o volume inicial?

2 – Num referencial S conhecem-se as seguintes componentes do tensor das deformações num ponto de um corpo:

$$d_{11} = d_{22} = -10^{-2} \;;\; d_{12} = 10^{-2} \;;\; d_{13} = 0 \;;\; d_{32} = 0$$

Sabendo que o corpo é constituído por um material homogéneo e isotrópico (elástico), de que se conhece o módulo de distorção, $G = 8 \times 10^4$ MPa e o módulo de Young, $E = 2 \times 10^5$ MPa e conhecendo-se ainda o invariante linear das tensões, de valor igual a -4×10^3 MPa , determine os elementos do tensor das tensões que caracteriza o estado de tensão no ponto no referencial das tensões principais.

3 – Um cilindro constituído por um material perfeitamente elástico, homogéneo e isotrópico é submetido a uma tensão de compressão, p, distribuída uniformemente nas suas bases sob dois tipos de condições fronteira na superfície lateral:

i) impede-se qualquer expansão transversal;

ii) a superfície lateral fica livre de qualquer tensão.

Sabendo que no primeiro caso o encurtamento é de 2/3 do encurtamento obtido no segundo caso, determine o coeficiente de Poisson do material e a pressão lateral exercida no primeiro caso.

4 – Considere os elementos do tensor das deformações num referencial $(0, x_1, x_2, x_3)$ que caracterizam o estado de deformação de um cilindro constituído por um material perfeitamente elástico, homogéneo e isotrópico.

$$d_{ij} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \times 10^{-2}$$

Determine os elementos do tensor das tensões correspondente no referencial dos eixos principais, sabendo que:

- uma compressão de 10 MPa, uniformemente distribuída na superfície lateral do cilindro origina uma extensão longitudinal de 3×10^{-5};
- a força uniformemente distribuída nas bases do cilindro necessária para anular aquela extensão é de 6 MPa.

5 – Considere o estado de tensão definido pelas seguintes componentes:

$$\sigma_x = 2xy \;;\; \sigma_y = 2x \;;\; \sigma_z = 2\nu x(1+y)$$

$$\tau_{xy} = -y^2 \;;\; \tau_{yz} = \tau_{zx} = 0$$

em que ν representa o coeficiente de Poisson de um meio elástico isotrópico, de módulo de elasticidade E.

Admitindo a não existência de forças mássicas, verifique que se trata de um estado de tensão possível, sob o ponto de vista estático e cinemático.

Resolução - FOLHA 8 - 1992/93

R-8.1

(1)

$$\underset{\sim}{\sigma} = \begin{bmatrix} 200 & 100 & 0 \\ 100 & 200 & 0 \\ 0 & 0 & -200 \end{bmatrix} (MPa) \qquad \begin{cases} E = 10^5 \ MPa \\ \nu = 0.25 \end{cases}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E \\ & 1/E & -\nu/E \\ \text{sim} & & 1/E \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \begin{bmatrix} 10^{-5} & -0.25\times10^{-5} & -0.25\times10^{-5} \\ & 10^{-5} & -0.25\times10^{-5} \\ \text{sim} & & 10^{-5} \end{bmatrix} \begin{bmatrix} 200 \\ 200 \\ -200 \end{bmatrix} = \begin{bmatrix} 2\times10^{-3} \\ 2\times10^{-3} \\ -3\times10^{-3} \end{bmatrix}$$

$$\frac{\Delta V}{V} = I_1(\varepsilon) = (2+2-3)\times10^{-3} = 10^{-3}$$

$$\bar{\sigma} = \frac{1}{3}\frac{E}{1-2\nu}\frac{\Delta V}{V} = \frac{1}{3}\frac{10^5}{1-2\times0.25}\times10^{-3} = 66.667 \ MPa$$

$$\hookrightarrow p = -66.667 \ MPa$$

Alternativa:

$$-p = \frac{1}{3}\frac{E}{1-2\nu}\frac{\Delta V}{V} = \cancel{\frac{1}{3}\frac{E}{1-2\nu}\frac{3(1-2\nu)}{E}}\bar{\sigma} = \bar{\sigma} = \frac{200+200-200}{3} = 66.6666 \ MPa$$

(2)

$$\underset{\sim}{d} = \begin{bmatrix} -0.01 & 0.01 & 0 \\ 0.01 & -0.01 & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \qquad \begin{aligned} G &= 80\,000 \ MPa \\ E &= 200\,000 \ MPa \\ G &= \frac{E}{2(1+\nu)} \Rightarrow \nu = 0.25 \end{aligned}$$

$$I(\sigma) = -4000$$

$$\begin{vmatrix} (-0.01-\varepsilon) & 0.01 & 0 \\ 0.01 & (-0.01-\varepsilon) & 0 \\ 0 & 0 & (d_{33}-\varepsilon) \end{vmatrix} = 0 \Rightarrow (d_{33}-\varepsilon)\left[(-0.01-\varepsilon)^2 - 0.01^2\right] = 0$$

$$\text{Soluções}\begin{cases} \varepsilon = d_{33} \\ \varepsilon^2 + \cancel{0.01^2} + 0.02\varepsilon - \cancel{0.01^2} = 0 \end{cases} \quad \begin{cases} \varepsilon = d_{33} \\ \varepsilon = 0 \\ \varepsilon = -0.02 \end{cases}$$

② (cont.)

No referencial principal: R - 8.2

$$\underset{\sim}{d} = \begin{bmatrix} d_{33} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.02 \end{bmatrix}$$

Constantes de Lamé
$$\begin{cases} \lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)} = 80\,000 \\ G = 80\,000 \end{cases} \quad \lambda + 2G = 240\,000$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \begin{bmatrix} 240000 & 80000 & 80000 \\ & 240\,000 & 80000 \\ \text{SIM} & & 240\,000 \end{bmatrix} \begin{bmatrix} d_{33} \\ 0 \\ -0.02 \end{bmatrix} = \begin{bmatrix} 240\,000\, d_{33} - 1600 \\ 80000\, d_{33} - 1600 \\ 80000\, d_{33} - 4800 \end{bmatrix}$$

$$\sigma_x + \sigma_y + \sigma_z = 400\,000\, d_{33} - 8000 = \underbrace{-4000}_{\text{Dado}} \Rightarrow d_{33} = 0.01$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} 800 & 0 & 0 \\ 0 & -800 & 0 \\ 0 & 0 & -4000 \end{bmatrix} \rightarrow \text{No referencial principal}$$

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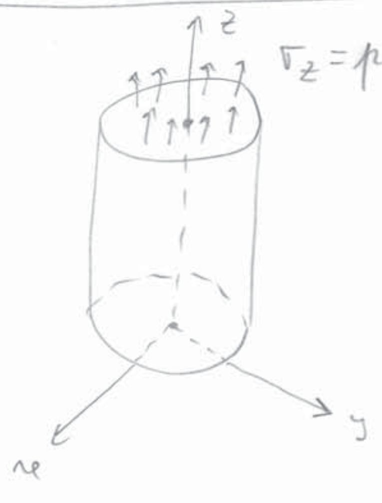
③



$\sigma_z = p$

Ⓐ $\begin{cases} \sigma_z^{(A)} = p \\ \sigma_x^{(A)} = \sigma_y^{(A)} = ? \end{cases}$ $\begin{cases} \varepsilon_z^{(A)} = ? \\ \varepsilon_x^{(A)} = \varepsilon_y^{(A)} = 0 \end{cases}$

Ⓑ $\begin{cases} \sigma_z^{(B)} = p \\ \sigma_x^{(B)} = \sigma_y^{(B)} = 0 \end{cases}$ $\begin{cases} \varepsilon_z^{(B)} = ? \\ \varepsilon_x^{(B)} = \varepsilon_y^{(B)} = ? \end{cases}$

$$\boxed{\varepsilon_z^{(A)} = \frac{2}{3}\, \varepsilon_z^{(B)}}$$

$$\begin{cases} \varepsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z \\ \varepsilon_y = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{\nu}{E}\sigma_z \\ \varepsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y + \frac{1}{E}\sigma_z \end{cases}$$

$$\begin{cases} \nu = ? \\ \sigma_x^{(A)} = \sigma_y^{(A)} = ? \end{cases}$$

③ (cont.) R-8.3

$$\varepsilon_x^{(A)} = 0 = \frac{1}{E}\sigma_x^{(A)} - \frac{\nu}{E}\sigma_x^{(A)} - \frac{\nu}{E}p \Rightarrow \sigma_x^{(A)}(1-\nu) = \nu p \Rightarrow \sigma_x^{(A)} = \frac{\nu}{1-\nu}p$$

$$\begin{cases} \varepsilon_z^{(B)} = \dfrac{1}{E}\sigma_z^{(B)} = \dfrac{p}{E} \\ \varepsilon_z^{(A)} = -\dfrac{\nu}{E}\sigma_x^{(A)} - \dfrac{\nu}{E}\sigma_y^{(A)} + \dfrac{1}{E}\sigma_z^{(A)} = \dfrac{p}{E} - \dfrac{2\nu}{E}\sigma_x^{(A)} = \dfrac{p}{E} - \dfrac{2\nu}{E}\dfrac{\nu}{1-\nu}p = \dfrac{p}{E}\left(1 - \dfrac{2\nu^2}{1-\nu}\right) \end{cases}$$

$$\varepsilon_z^{(A)} = \frac{2}{3}\varepsilon_z^{(B)} \Rightarrow \frac{\cancel{p}}{\cancel{E}}\left(1-\frac{2\nu^2}{1-\nu}\right) = \frac{2}{3}\frac{\cancel{p}}{\cancel{E}} \Rightarrow \frac{1}{3} = \frac{2\nu^2}{1-\nu} \Rightarrow 6\nu^2+\nu-1=0 \quad \begin{cases}\nu = \frac{1}{3} \\ \cancel{\nu = -\frac{1}{2}}\end{cases}$$

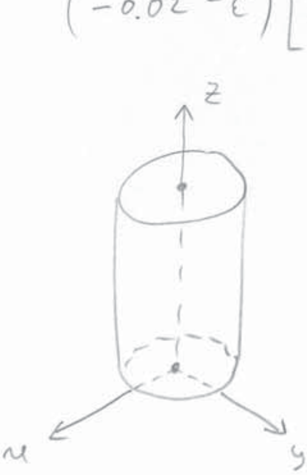
$$\boxed{\nu = \frac{1}{3}} \Rightarrow \boxed{\sigma_x^{(A)} = \frac{\nu}{1-\nu}p = \frac{1/3}{2/3}p = 0.5\,p}$$

---

④

$$\underset{\sim}{d} = \begin{bmatrix} 0.01 & 0.01 & 0 \\ 0.01 & 0.01 & 0 \\ 0 & 0 & -0.02 \end{bmatrix} \rightarrow \begin{vmatrix} (0.01-\varepsilon) & 0.01 & 0 \\ 0.01 & (0.01-\varepsilon) & 0 \\ 0 & 0 & (-0.02-\varepsilon) \end{vmatrix} = 0$$

$$(-0.02-\varepsilon)\left[(0.01-\varepsilon)^2 - 0.01^2\right] = 0 \Rightarrow (-0.02-\varepsilon)(\varepsilon^2 - 0.02\varepsilon) = 0 \quad \begin{cases}\varepsilon = 0.02 \\ \varepsilon = 0 \\ \varepsilon = -0.02\end{cases}$$



$$\left.\begin{array}{l} \sigma_x^{(A)} = \sigma_y^{(A)} = -10\,MPa \\ \sigma_z^{(A)} = 0 \end{array}\right\} \Rightarrow \varepsilon_z^{(A)} = 3\times 10^{-5}$$

$$\left.\begin{array}{l} \sigma_x^{(B)} = \sigma_y^{(B)} = 0 \\ \sigma_z^{(B)} = 6\,MPa \end{array}\right\} \Rightarrow \varepsilon_z^{(B)} = 3\times 10^{-5}$$

4 (cont.) $\varepsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y + \frac{1}{E}\sigma_z$ R-8.4

$$\varepsilon_z^{A} = \varepsilon_z^{B} \Rightarrow -\frac{2\nu}{E}(-10) = \frac{1}{E}6 \Rightarrow 20\nu = 6 \Rightarrow \boxed{\nu = 0.3}$$

$$\varepsilon_z^{B} = 3\times10^{-5} = \frac{1}{E}6 \Rightarrow \boxed{E = 200\,000\ MPa}$$

Constantes de Lamé
$$\begin{cases} \lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)} = 115\,385 \\ G = \dfrac{E}{2(1+\nu)} = 76\,923 \end{cases}$$

$\lambda + 2G = 269\,231$

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix} = \begin{bmatrix}269231 & 115385 & 115385\\ & 269231 & 115385\\ \text{Sim} & & 269231\end{bmatrix}\begin{bmatrix}0.02\\ 0\\ -0.02\end{bmatrix} = \begin{bmatrix}3076.92\\ 0\\ -3076.92\end{bmatrix} \Rightarrow \underset{\sim}{\tau} = \begin{bmatrix}3076.92 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -3076.92\end{bmatrix}$$

---

5 $\sigma_x = 2xy$ $\sigma_y = 2x$ $\sigma_z = 2\nu x(1+y)$

$\tau_{xy} = -y^2$ $\tau_{yz} = 0$ $\tau_{zx} = 0$

Tem de verificar as equações de equilíbrio indefinido
(na ausência de forças de massa → $\tau_{ij,j} = 0$)

$$\begin{cases} \dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{xy}}{\partial y} + \dfrac{\partial\tau_{xz}}{\partial z} = 0 \\ \dfrac{\partial\tau_{yx}}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} + \dfrac{\partial\tau_{yz}}{\partial z} = 0 \\ \dfrac{\partial\tau_{zx}}{\partial x} + \dfrac{\partial\tau_{zy}}{\partial y} + \dfrac{\partial\sigma_z}{\partial z} = 0 \end{cases} \quad \begin{cases} 2y - 2y + 0 = 0 \text{ (OK)} \\ 0 + 0 + 0 = 0 \text{ (OK)} \\ 0 + 0 + 0 = 0 \text{ (OK)} \end{cases}$$

(5) (cont.)

R-8.5

Estado de deformação associado:

$$\begin{cases} \varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right) = \frac{1}{E}\left[2xy - 2\nu x - 2\nu^2 x(1+y)\right] \\ \varepsilon_y = \frac{1}{E}\left(-\nu\sigma_x + \sigma_y - \nu\sigma_z\right) = \frac{1}{E}\left[-2\nu xy + 2x - 2\nu^2 x(1+y)\right] \\ \varepsilon_z = \frac{1}{E}\left(-\nu\sigma_x - \nu\sigma_y + \sigma_z\right) = \frac{1}{E}\left[-2\nu xy - 2\nu x + 2\nu x(1+y)\right] = 0 \end{cases}$$

$$\begin{cases} \gamma_{yz} = \frac{1}{G}\tau_{yz} = 0 \\ \gamma_{zx} = \frac{1}{G}\tau_{zx} = 0 \\ \gamma_{xy} = \frac{1}{G}\tau_{xy} = -\frac{y^2}{G} \end{cases}$$

Tem de verificar as equações de compatibilidade:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial \varepsilon_x}{\partial y} = \frac{1}{E}\left(2x - 2\nu^2 x\right) \rightarrow \frac{\partial^2 \varepsilon_x}{\partial y^2} = 0$$

$$\frac{\partial \varepsilon_y}{\partial x} = \frac{1}{E}\left[-2\nu y + 2 - 2\nu^2(1+y)\right] \rightarrow \frac{\partial^2 \varepsilon_y}{\partial x^2} = 0$$

$$\frac{\partial \gamma_{xy}}{\partial x} = 0 \rightarrow \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

$0 + 0 = 0$ (OK)

As restantes equações de compatibilidade verificam-se porque $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$ e ε_x, ε_y e γ_{xy} não dependem de z.