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Métodos Numéricos de Análise Estrutural

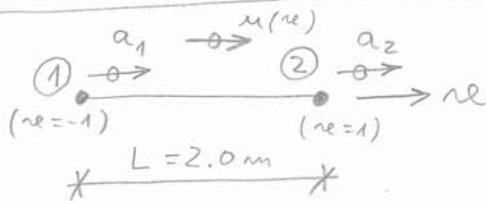
5º ano da Licenciatura em Engenharia Civil

Faculdade de Engenharia da Universidade do Porto - Portugal

Novembro 1995

$E$  → módulo de elasticidade  
 $\sigma$  → tensão  
 $\epsilon$  → deformação  
 $D$  → matriz de elasticidade  
 $u$  → deslocamento  
 $a$  → deslocamento nodal  
 $n$  → número de nós do elemento

$K$  → matriz de rigidez  
 $F$  → vetor solicitação  
 $N$  → função de forma  
 $p$  → ação exterior distribuída por unidade de comprimento  
 $A$  → área da secção transversal  
 $L$  → comprimento da barra  
 $\nu$  → coeficiente de Poisson

Alvaro  
AzevedoNovembro  
1995

Supõe-se que o campo de deslocamentos  $u(x)$  é linear.

$$u(x) = \frac{a_2 - a_1}{2} x + \frac{a_1 + a_2}{2}$$

$$\begin{aligned} x = -1 &\Rightarrow u = a_1 \\ x = 1 &\Rightarrow u = a_2 \end{aligned}$$

↳  $u$  é uma função linear de  $x$ , supondo que  $a_1$  e  $a_2$  possuem valores conhecidos.

Apresenta-se em seguida um formato mais apropriado para um tratamento matricial:

$$u(x) = N_1(x) a_1 + N_2(x) a_2 = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Para  $x = -1$ , tem que se verificar  $u = a_1$ ; então  $\begin{cases} N_1(-1) = 1 \\ N_2(-1) = 0 \end{cases}$

Para  $x = 1$ , tem que se verificar  $u = a_2$ ; então  $\begin{cases} N_1(1) = 0 \\ N_2(1) = 1 \end{cases}$

Função  $N_1(xe)$  que verifica  $\begin{cases} xe = -1 \Rightarrow N_1(-1) = 1 \\ xe = 1 \Rightarrow N_1(1) = 0 \end{cases} \rightarrow N_1(xe) = -\frac{1}{2} xe + \frac{1}{2}$

Função  $N_2(xe)$  que verifica  $\begin{cases} xe = -1 \Rightarrow N_2(-1) = 0 \\ xe = 1 \Rightarrow N_2(1) = 1 \end{cases} \rightarrow N_2(xe) = \frac{1}{2} xe + \frac{1}{2}$

$$u(xe) = \left(-\frac{1}{2} xe + \frac{1}{2}\right) a_1 + \left(\frac{1}{2} xe + \frac{1}{2}\right) a_2$$

$m \rightarrow$  número de nós do elemento

$$u(xe) = \begin{bmatrix} -\frac{1}{2} xe + \frac{1}{2} & \frac{1}{2} xe + \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Leftrightarrow \begin{matrix} \boxed{u = N a} \\ \sim \quad \sim \quad \sim \\ \downarrow \quad \downarrow \quad \downarrow \\ (1 \times 1) \quad (1 \times m) \quad (m \times 1) \end{matrix}$$

$$\boxed{N = \begin{bmatrix} -\frac{1}{2} xe + \frac{1}{2} & \frac{1}{2} xe + \frac{1}{2} \end{bmatrix}}$$

$$\underline{\underline{E}} = \frac{d \underline{\underline{u}}}{d \underline{\underline{x}}}$$

$$\underline{\underline{E}} = \left[ \frac{d}{d \underline{\underline{x}}} \right] \begin{bmatrix} u \end{bmatrix}$$

$$\underline{\underline{L}} = \left[ \frac{d}{d \underline{\underline{x}}} \right]$$

$\hookrightarrow$  operador diferencial

$$\begin{matrix} \underline{\underline{E}} = \underline{\underline{L}} \underline{\underline{u}} \\ \sim \quad \sim \quad \sim \\ \quad \quad \uparrow \\ \quad \quad \underline{\underline{u}} = \underline{\underline{N}} \underline{\underline{a}} \\ \quad \quad \sim \quad \sim \end{matrix}$$

$$\underline{\underline{E}} = \underline{\underline{L}} \underline{\underline{N}} \underline{\underline{a}}$$

$$\boxed{\underline{\underline{E}} = \underline{\underline{B}} \underline{\underline{a}}}$$

$$\boxed{\underline{\underline{B}} = \underline{\underline{L}} \underline{\underline{N}}}$$

$$\underset{\sim}{\epsilon} = \underset{\sim}{L} \underset{\sim}{N} \underset{\sim}{a}$$

$$\epsilon = \begin{bmatrix} \frac{\partial}{\partial u} \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} \frac{\partial N_1}{\partial u} & \frac{\partial N_2}{\partial u} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\underset{\sim}{\epsilon} = \underset{\sim}{B} \underset{\sim}{a}$$

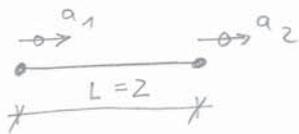
$$\underset{\sim}{B} = \begin{bmatrix} \frac{\partial N_1}{\partial u} & \frac{\partial N_2}{\partial u} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(Ver as funções  $N_1(u)$  e  $N_2(u)$  na página anterior.)

$$\underset{\sim}{\epsilon} = \underset{\sim}{B} \underset{\sim}{a} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\frac{1}{2} a_1 + \frac{1}{2} a_2 = \frac{a_2 - a_1}{2}$$

$$(\Delta L = a_2 - a_1)$$

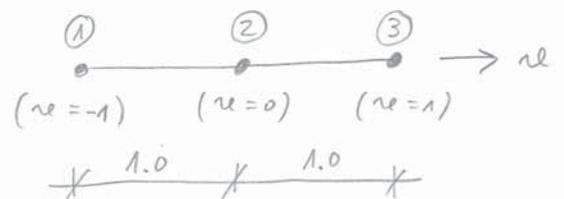
$$\epsilon = \frac{a_2 - a_1}{2} = \frac{\Delta L}{L} \rightarrow$$



Nota:  $\epsilon$  não depende de  $u$ , portanto a deformação em todos os pontos deste elemento finito é constante.

### PROBLEMA ①

Calcular a matriz  $\underset{\sim}{B}$  correspondente ao seguinte elemento finito de três nós:





trabalho interno = trabalho externo

$$\int_V \tilde{\epsilon}^T \tilde{\sigma} dV = \int_L \tilde{u}^T p d\tilde{x}$$

$$\int_V \tilde{\epsilon}^T \tilde{D} \tilde{\epsilon} dV = \int_L \tilde{a}^T \tilde{N}^T p d\tilde{x}$$

$$\int_V \tilde{a}^T \tilde{B}^T \tilde{D} \tilde{B} \tilde{a} dV = \int_L \tilde{a}^T \tilde{N}^T p d\tilde{x}$$

Notas:

$$\tilde{D} = [E]$$

$$\tilde{\sigma} = \tilde{D} \tilde{\epsilon}$$

$$\tilde{u} = \tilde{N} \tilde{a} \Leftrightarrow \tilde{u}^T = \tilde{a}^T \tilde{N}$$

$$\tilde{\epsilon} = \tilde{B} \tilde{a} \Leftrightarrow \tilde{\epsilon}^T = \tilde{a}^T \tilde{B}$$

$$\tilde{a}^T \left[ \int_V \tilde{B}^T \tilde{D} \tilde{B} dV \right] \tilde{a} = \tilde{a}^T \left[ \int_L \tilde{N}^T p d\tilde{x} \right]$$

$$\underbrace{\left[ \int_V \tilde{B}^T \tilde{D} \tilde{B} dV \right]}_{\tilde{K}} \tilde{a} = \underbrace{\left[ \int_L \tilde{N}^T p d\tilde{x} \right]}_{\tilde{F}}$$

$$\tilde{K} \tilde{a} = \tilde{F}$$

$$\tilde{K} = \int_V \tilde{B}^T \tilde{D} \tilde{B} dV$$

→ matriz de rigidez

$$\tilde{F} = \int_L \tilde{N}^T p d\tilde{x}$$

→ vetor solicitações

Elemento finito com dois nós (nós atrás):

MEF-5

$$\tilde{K} = \int_L \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} E \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} A \, du =$$

$$= \int_{-1}^{+1} \begin{bmatrix} \frac{EA}{4} & -\frac{EA}{4} \\ -\frac{EA}{4} & \frac{EA}{4} \end{bmatrix} du = \begin{bmatrix} \frac{EA}{2} & -\frac{EA}{2} \\ -\frac{EA}{2} & \frac{EA}{2} \end{bmatrix}$$

Nota:  $L=2$

$$K_{ij} = \pm \frac{EA}{L}$$

$$\tilde{F} = \int_L \begin{bmatrix} -\frac{1}{2}u + \frac{1}{2} \\ \frac{1}{2}u + \frac{1}{2} \end{bmatrix} p \, du = \int_{-1}^{+1} \begin{bmatrix} \frac{1-u}{2} p \\ \frac{1+u}{2} p \end{bmatrix} du = \begin{bmatrix} p \\ p \end{bmatrix}$$

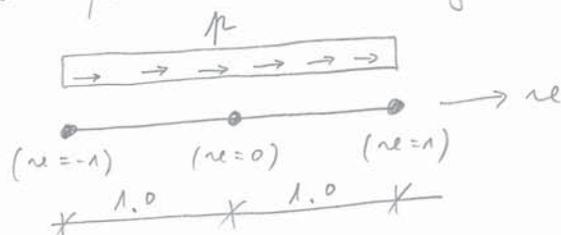
Nota: numa viga de comprimento  $L$ ,  $\tilde{K}$  e  $\tilde{F}$  apresentam as seguintes expressões:

$$\tilde{K} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \quad \tilde{F} = \begin{bmatrix} \frac{pL}{2} \\ \frac{pL}{2} \end{bmatrix}$$

Coincidem com as que foram atrás obtidas para um elemento finito com  $L=2$ .

### PROBLEMA (2)

Calcular a matriz de rigidez e o vector solicitações correspondentes ao seguinte elemento finito:

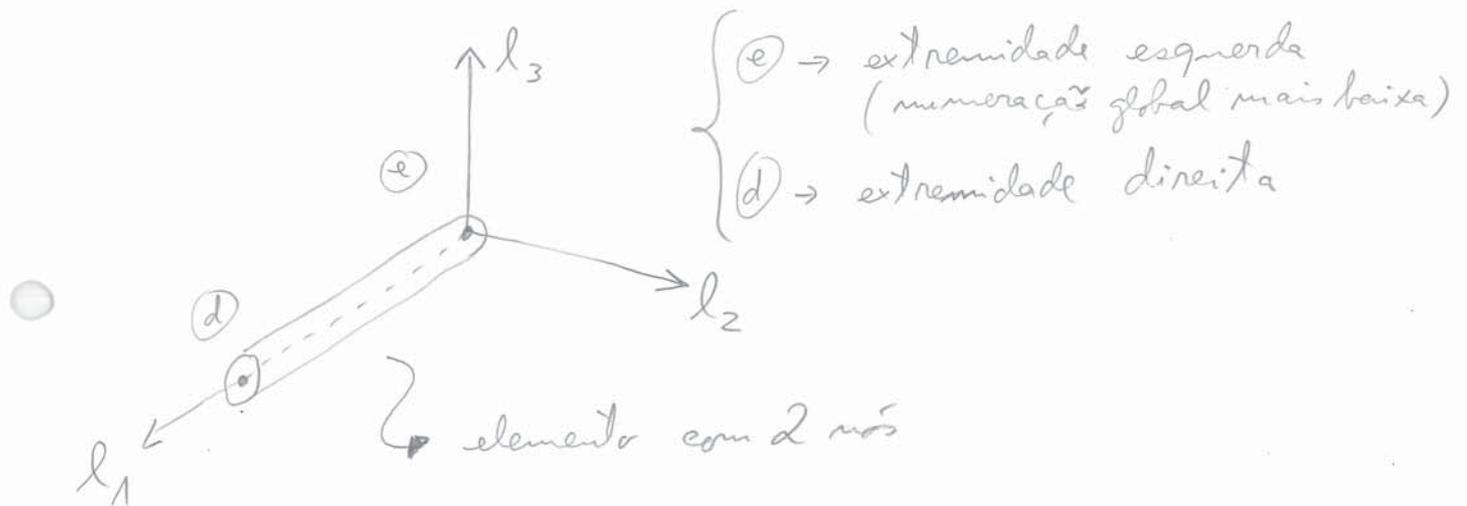


$$\begin{cases} E \rightarrow \text{constantes} \\ A \rightarrow \end{cases}$$

Fim da 1ª aula

Matriz de transformação 3D (ver TC-1 a TC-10)

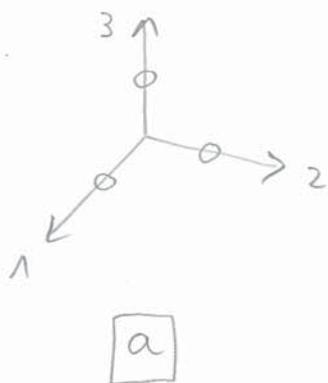
Matriz de rigidez de um elemento de treliça 3D



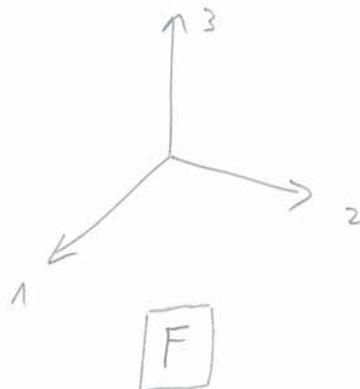
→ Considerar a matriz de transformação para  $d=0$

- $\left\{ \begin{array}{l} \text{Barras não verticais} \Rightarrow l_2 \text{ perpendicular ao plano ver-} \\ \text{tical que contém a barra} \\ \text{Barras verticais} \Rightarrow l_2 \equiv g_2 \end{array} \right.$

Deslocamentos



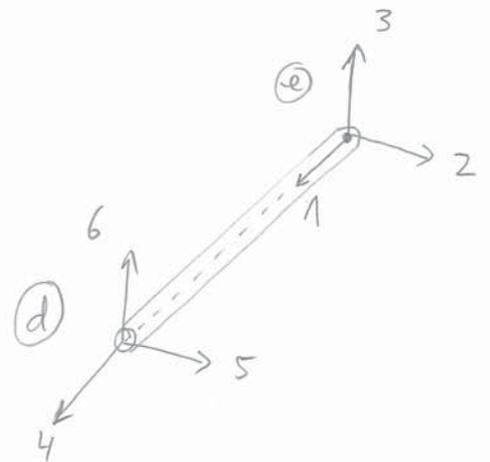
Forças



Características da barra  $\left\{ \begin{array}{l} A \rightarrow \text{área da seção transversal} \\ L \rightarrow \text{comprimento da barra} \\ E \rightarrow \text{módulo de elasticidade} \end{array} \right. \text{MEF-7}$

Matriz de rigidez da barra no referencial local:

$$K_l = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$



$$\tilde{u}_l = T \tilde{u}_g$$

$$\tilde{F}_g = T^T \tilde{F}_l$$

$$\tilde{F}_l = K_l \tilde{a}_l$$

$$\tilde{a}_l = T \tilde{a}_g$$

$$\tilde{F}_g = \left( T^T K_l T \right) \tilde{a}_g$$

$$\tilde{F}_g = K_g \tilde{a}_g$$

$$K_g = T^T K_l T$$

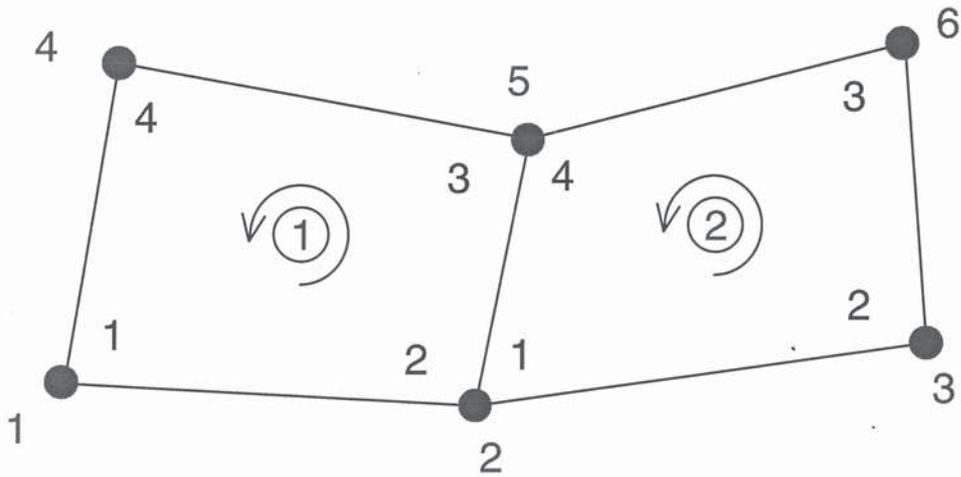
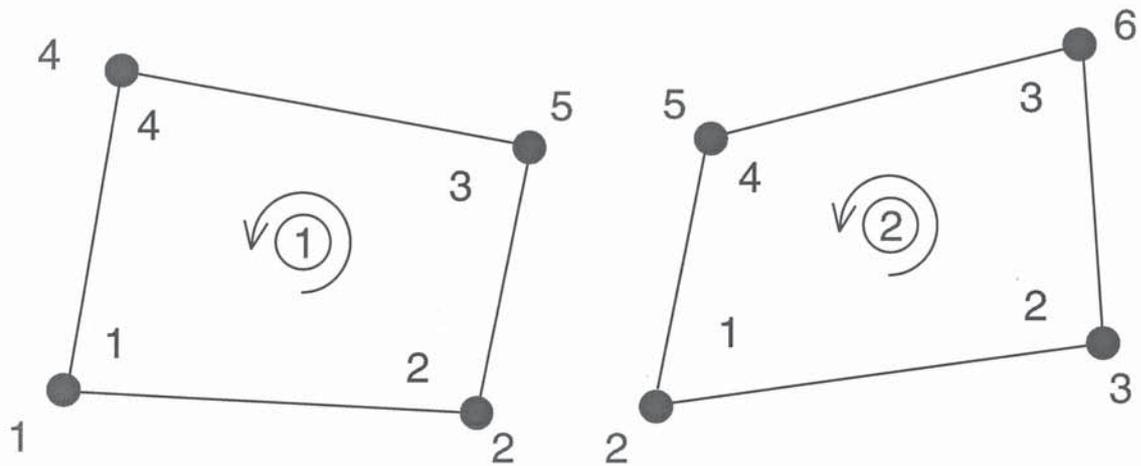
→ matriz de rigidez no referencial geral

Nota:  $T^T = T^{-1}$   
(a matriz de transformação  $T$  é ortogonal)

$$K_g \tilde{a}_g = \tilde{F}_g$$

Fim da 2ª aula

# ASSEMBLAGEM DE MATRIZES DE RIGIDEZ LOCAIS NA MATRIZ DE RIGIDEZ GLOBAL



### Matriz de rigidez do elemento 1 no referencial geral

- apenas é considerada uma variável por nó

	1	2	3	4	5	6
1	1.11	1.12		1.14	1.13	
2	1.21	1.22		1.24	1.23	
3						
4	1.41	1.42		1.44	1.43	
5	1.31	1.32		1.34	1.33	
6						

### Matriz de rigidez do elemento 2 no referencial geral

- apenas é considerada uma variável por nó

	1	2	3	4	5	6
1						
2		2.11	2.12		2.14	2.13
3		2.21	2.22		2.24	2.23
4						
5		2.41	2.42		2.44	2.43
6		2.31	2.32		2.34	2.33

### Matriz de rigidez global no referencial geral

- apenas é considerada uma variável por nó

	1	2	3	4	5	6
1	1.11	1.12		1.14	1.13	
2	1.21	1.22	2.11	1.24	1.23	2.13
3						
4	1.41	1.42	2.21	1.44	1.43	2.23
5	1.31	1.32		1.34	1.33	
6		2.41	2.42		2.44	2.43
		2.31	2.32		2.34	2.33

# Grãos de liberdade prescritos

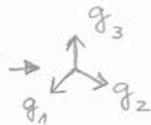
MEF-12

(assentamentos de apoio)

Apoios fixos  $\rightarrow$  são tratados como assentamentos de apoio de valor nulo.

$\left\{ \begin{array}{l} L \rightarrow \text{grãos de liberdade não prescritos (livres)} \\ F \rightarrow \text{grãos de liberdade prescritos (fixos)} \end{array} \right.$

$$\begin{matrix} K & a & = & F \\ \sim & \sim & & \sim \end{matrix}$$

$\rightarrow$  referencial geral  $\rightarrow$  

$\rightarrow$  matriz de rigidez global (relaciona todos os grãos de liberdade da estrutura)

	Livre	Fixo	
Livre	$K_{LL}$	$K_{LF}$	$\times \begin{matrix} a_L \\ a_F \end{matrix} = \begin{matrix} F_L \\ F_F \end{matrix} + \begin{matrix} 0 \\ R_F \end{matrix}$
Fixo	$K_{FL}$	$K_{FF}$	

$R_F \rightarrow$  reações nos grãos de liberdade prescritos

$a_L$  e  $R_F \rightarrow$  incógnitas

$$K_{LL} a_L + K_{LF} a_F = F_L$$

Calculando previamente  $(F'_L = F_L - K_{LF} a_F)$  pode-se considerar:

$$K_{LL} a_L = F'_L \left\{ \begin{array}{l} \rightarrow \text{sistema de equações lineares que pode ser resolvido por um método direto ou iterativo.} \\ \rightarrow \text{obtem-se assim os deslocamentos } a_L \end{array} \right.$$

$$K_{FL} a_L + K_{FF} a_F = F_F + R_F$$

$$R_F = K_{FL} a_L + K_{FF} a_F - F_F \rightarrow \text{obtem-se assim as reações}$$

$\rightarrow$  Recorrendo a este algoritmo, os grãos de liberdade prescritos não contribuem para a dimensão do sistema de equações

## Faseamento da resolução:

Para cada barra:

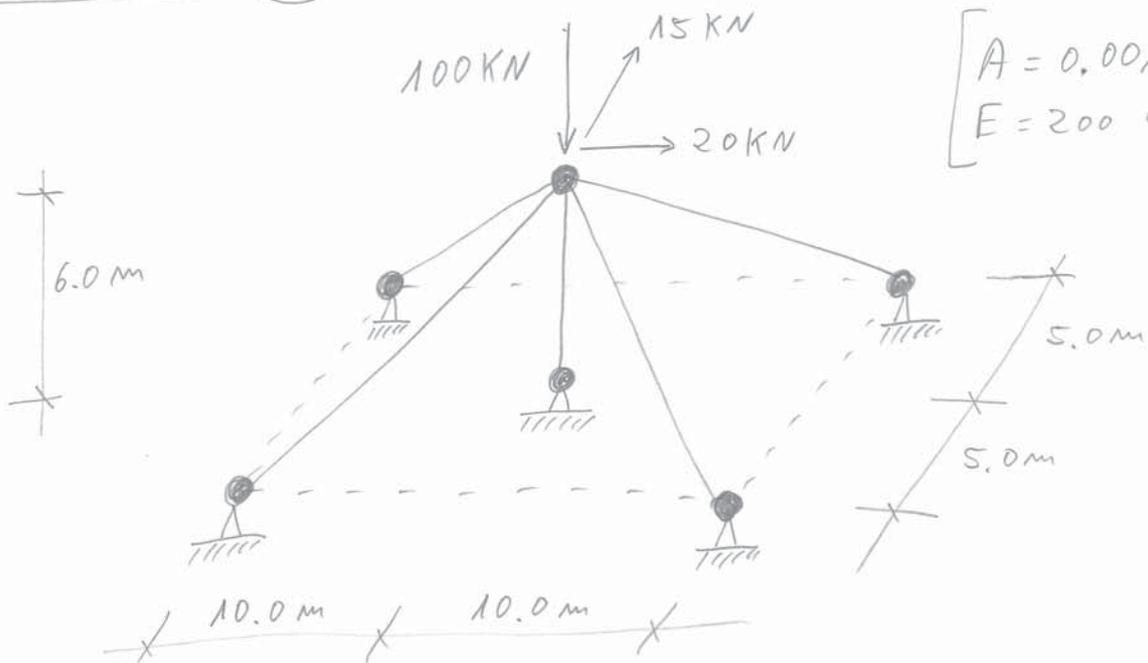
- (A) - calcular a matriz de transformações
- (B) - calcular a matriz de rigidez no referencial local
- (C) - calcular a matriz de rigidez no referencial global
- (D) - montar a matriz de rigidez global

- Resolver o sistema de equações entrando em consideração com os deslocamentos prescritos
  - Calcular os esforços nas barras
-

PROBLEMA

(3)

MEF-14



Calcular os esforços nas barras:

- Resolver com base em  $\tilde{K}_g = \tilde{T}^T \tilde{K} \tilde{T}$
- Resolver com um programa de cálculo automático.
- Aumentar progressivamente a área das barras inclinadas até surgirem problemas numéricos.

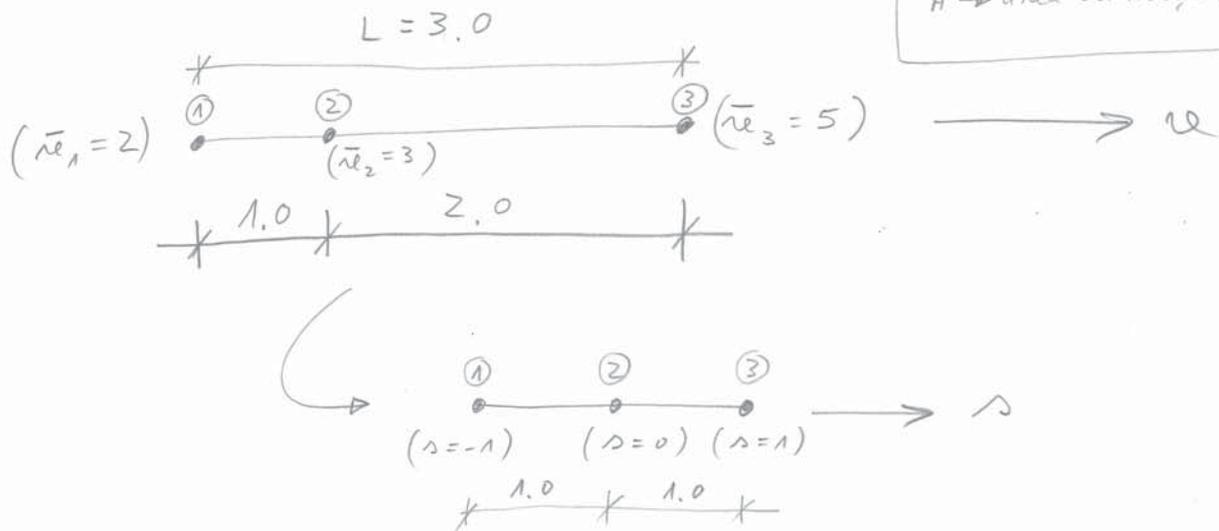
Fim da 3ª aula

4: aula → apresentação do FEMIX

MEF-15

5: aula

Calcular a matriz de rigidez do seguinte elemento unidimensional:



$E \rightarrow$  módulo de elasticidade  
 $A \rightarrow$  área da seção transversal

$$\tilde{K} = \int_V (\tilde{B}^T \tilde{D} \tilde{B}) dV = \int_L (\tilde{B}^T \tilde{D} \tilde{B}) A du$$

$(3 \times 1)$   $(1 \times 1)$   $(1 \times 3)$

$$\tilde{B} = \begin{bmatrix} \frac{dN_1}{du} & \frac{dN_2}{du} & \frac{dN_3}{du} \end{bmatrix}$$

$$\tilde{D} = [E]$$

$$\begin{cases} N_1(s) = \frac{1}{2}s^2 - \frac{1}{2}s \\ N_2(s) = -s^2 + 1 \\ N_3(s) = \frac{1}{2}s^2 + \frac{1}{2}s \end{cases} \begin{cases} \frac{dN_1}{ds} = s - \frac{1}{2} \\ \frac{dN_2}{ds} = -2s \\ \frac{dN_3}{ds} = s + \frac{1}{2} \end{cases}$$

$$\frac{dN_i}{ds} = \frac{dN_i}{du} \frac{du}{ds} \Rightarrow \frac{dN_i}{du} = \left[ \frac{du}{ds} \right]^{-1} \frac{dN_i}{ds} \quad (i=1, \dots, 3)$$

$$\begin{cases} u = \underset{\sim}{N} \underset{\sim}{a} = N_1 a_1 + N_2 a_2 + N_3 a_3 \\ u = \underset{\sim}{N} \underset{\sim}{\bar{u}} = N_1 \bar{u}_1 + N_2 \bar{u}_2 + N_3 \bar{u}_3 \end{cases}$$

$$\frac{du}{ds} = \frac{d}{ds} \left( \underset{\sim}{N} \underset{\sim}{\bar{u}} \right) = \frac{d\underset{\sim}{N}}{ds} \underset{\sim}{\bar{u}} = \frac{dN_1}{ds} \bar{u}_1 + \frac{dN_2}{ds} \bar{u}_2 + \frac{dN_3}{ds} \bar{u}_3$$

$$\mathcal{J} = \frac{du}{ds} = \left(s - \frac{1}{2}\right) 2 + (-2s) 3 + \left(s + \frac{1}{2}\right) 5 = s + \frac{3}{2}$$

$$\left[ \frac{du}{ds} \right]^{-1} = \frac{1}{s + \frac{3}{2}}$$

$$\frac{dN_1}{dne} = \left[ \frac{dne}{ds} \right]^{-1} \frac{dN_1}{ds} = \frac{s - \frac{1}{2}}{s + \frac{3}{2}}$$

$$\frac{dN_2}{dne} = \left[ \frac{dne}{ds} \right]^{-1} \frac{dN_2}{ds} = \frac{-2s}{s + \frac{3}{2}}$$

$$\frac{dN_3}{dne} = \left[ \frac{dne}{ds} \right]^{-1} \frac{dN_3}{ds} = \frac{s + \frac{1}{2}}{s + \frac{3}{2}}$$

$$\underset{\sim}{B} = \begin{bmatrix} \frac{s - \frac{1}{2}}{s + \frac{3}{2}} & -\frac{2s}{s + \frac{3}{2}} & \frac{s + \frac{1}{2}}{s + \frac{3}{2}} \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} [E] \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} Ea^2 & Eab & Eac \\ Eab & Eb^2 & Ebc \\ Eac & Ebc & Ec^2 \end{bmatrix}$$

(3x1) (1x1) (1x3)

$$\underset{\sim}{K} = \int_L^{+1} (\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B}) A dne = \int_{-1}^{+1} (\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B}) \left[ \frac{dne}{ds} \right] A ds$$

$$K_{11} = \int_{-1}^{+1} (Ea^2) \left[ \frac{dne}{ds} \right] A ds = EA \int_{-1}^{+1} \left( \frac{s - \frac{1}{2}}{s + \frac{3}{2}} \right)^2 \left( s + \frac{3}{2} \right) ds$$

$$K_{11} = EA \int_{-1}^{+1} \frac{s^2 - s + \frac{1}{4}}{s + \frac{3}{2}} ds$$

$$K_{11} = 1.437\ 751\ 650\ EA \quad \left( \begin{array}{l} \text{ver a página} \\ \text{seguinte} \end{array} \right)$$

N. Pontos Gauss	$K_{11}$
1	0.333 333 EA
2	1.260 870 EA
3	1.410 774 EA
4	1.433 722 EA
5	1.437 155 EA
6	1.437 664 EA
7	1.437 739 EA
...	
∞	

$$\int \frac{s^2}{s + \frac{3}{2}} ds = \frac{(s + \frac{3}{2})^2}{2} - 3(s + \frac{3}{2}) + \frac{9}{4} \ln(s + \frac{3}{2})$$

$$\int \frac{s}{s + \frac{3}{2}} ds = s - \frac{3}{2} \ln(s + \frac{3}{2})$$

$$\int \frac{1}{s + \frac{3}{2}} ds = \ln(s + \frac{3}{2})$$

$$\bullet \quad \frac{s^2}{2} + \frac{3}{2}s + \frac{9}{8} - 3s + \frac{9}{2} + \frac{9}{4} \ln(s + \frac{3}{2}) - s + \frac{3}{2} \ln(s + \frac{3}{2}) + \frac{1}{4} \ln(s + \frac{3}{2})$$

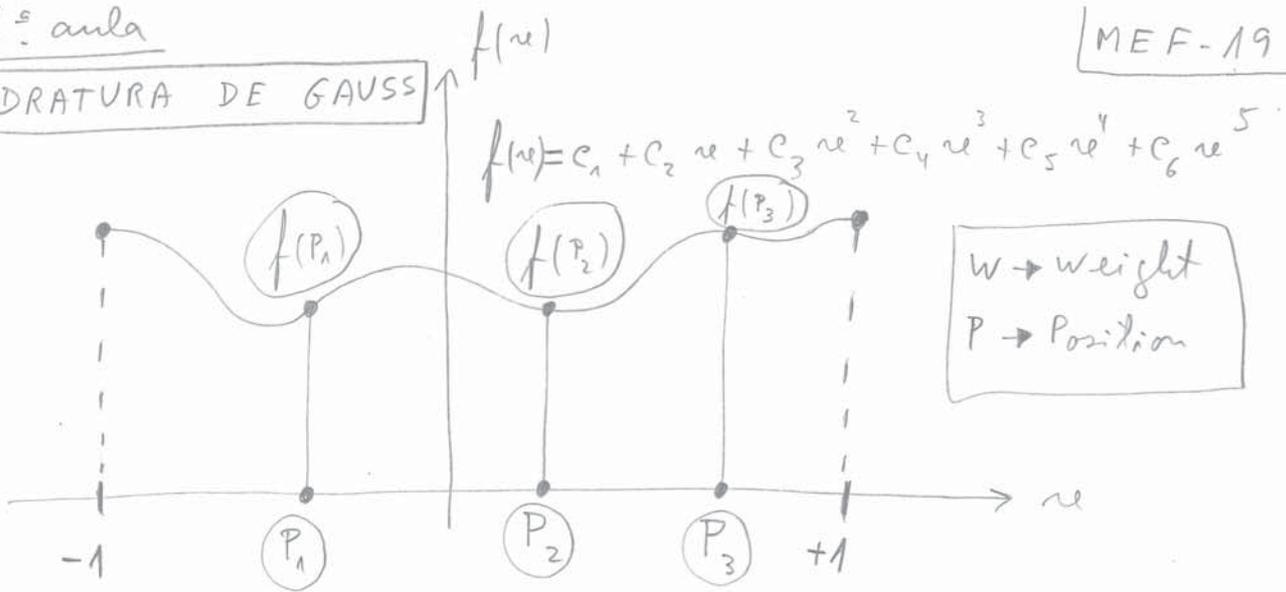
$$= \left[ \frac{s^2}{2} - \frac{5}{2}s + \frac{45}{8} + 4 \ln(s + \frac{3}{2}) \right]_{-1}^1 =$$

$$= \left[ \frac{1}{2} - \frac{5}{2} + \frac{45}{8} + 4 \ln\left(\frac{5}{2}\right) \right] - \left[ \frac{1}{2} + \frac{5}{2} + \frac{45}{8} + 4 \ln\left(\frac{1}{2}\right) \right] =$$

$$\bullet \quad = -5 + 4 \ln\left(\frac{5}{2}\right) - 4 \ln\left(\frac{1}{2}\right) = 1.437 \quad 751 \quad 650$$

Fim da 5ª aula

## QUADRATURA DE GAUSS



$$I = \int_{-1}^{+1} f(x) dx = W_1 f(P_1) + W_2 f(P_2) + W_3 f(P_3)$$

$$I = \int_{-1}^{+1} f(x) dx = \int_{-1}^{+1} (c_1 + c_2x + c_3x^2 + c_4x^3 + c_5x^4 + c_6x^5) dx =$$

$$= \left[ c_1x + c_2 \frac{x^2}{2} + c_3 \frac{x^3}{3} + c_4 \frac{x^4}{4} + c_5 \frac{x^5}{5} + c_6 \frac{x^6}{6} \right]_{-1}^{+1} =$$

Os termos com expoente par desaparecem:

$$= \left( c_1 + \frac{c_3}{3} + \frac{c_5}{5} \right) - \left( -c_1 - \frac{c_3}{3} - \frac{c_5}{5} \right) = 2c_1 + \frac{2}{3}c_3 + \frac{2}{5}c_5$$

$$I = \int_{-1}^{+1} f(x) dx = W_1 (c_1 + c_2 P_1 + c_3 P_1^2 + c_4 P_1^3 + c_5 P_1^4 + c_6 P_1^5) +$$

$$+ W_2 (c_1 + c_2 P_2 + c_3 P_2^2 + c_4 P_2^3 + c_5 P_2^4 + c_6 P_2^5) +$$

$$+ W_3 (c_1 + c_2 P_3 + c_3 P_3^2 + c_4 P_3^3 + c_5 P_3^4 + c_6 P_3^5) =$$

$$= (w_1 + w_2 + w_3) e_1 +$$

$$+ (w_1 P_1 + w_2 P_2 + w_3 P_3) e_2 +$$

$$+ (w_1 P_1^2 + w_2 P_2^2 + w_3 P_3^2) e_3 +$$

$$+ (w_1 P_1^3 + w_2 P_2^3 + w_3 P_3^3) e_4 +$$

$$+ (w_1 P_1^4 + w_2 P_2^4 + w_3 P_3^4) e_5 +$$

$$+ (w_1 P_1^5 + w_2 P_2^5 + w_3 P_3^5) e_6$$

$$= \underbrace{+0e_2}_{+0e_2} + \underbrace{\frac{2}{3}e_3}_{+0e_4} + \underbrace{\frac{2}{5}e_5}_{+0e_6}$$

$$\left( \forall e_1, e_2, e_3, e_4, e_5, e_6 \right)$$

$$w_1 + w_2 + w_3 = 2$$

$$w_1 P_1 + w_2 P_2 + w_3 P_3 = 0$$

$$w_1 P_1^2 + w_2 P_2^2 + w_3 P_3^2 = \frac{2}{3}$$

$$w_1 P_1^3 + w_2 P_2^3 + w_3 P_3^3 = 0$$

$$w_1 P_1^4 + w_2 P_2^4 + w_3 P_3^4 = \frac{2}{5}$$

$$w_1 P_1^5 + w_2 P_2^5 + w_3 P_3^5 = 0$$

incógnitas:

$(w_1, w_2, w_3,$   
 $P_1, P_2, P_3)$

→ sistema de 6 equações não lineares a 6 incógnitas

$$\rightarrow \text{Solução} \begin{cases} w_1 = 0.55555 & 55556 \\ w_2 = 0.88888 & 88889 \\ w_3 = 0.55555 & 55556 \end{cases} \parallel \begin{cases} P_1 = -0.77459 & 66692 \\ P_2 = 0.0 \\ P_3 = 0.77459 & 66692 \end{cases}$$

$$w_1 = 5/9 \quad ; \quad w_2 = 8/9$$

$$P_3 = \sqrt{3}/\sqrt{5}$$

## Expressões gerais da quadratura de Gauss:

$$\begin{cases} m \rightarrow \text{número de pontos de Gauss} \\ m = 2n \end{cases}$$

$$f(x) = c_1 + c_2 x + c_3 x^2 + \dots + c_m x^{m-1}$$

$$I = \int_{-1}^{+1} f(x) dx = 2c_1 + \frac{2}{3}c_3 + \frac{2}{5}c_5 + \dots$$

$$I = \sum_{\substack{j=1 \\ (\text{ímpar})}}^{m-1} \frac{2}{j} c_j$$

Sistema de  $m$  equações não lineares com  $m$  incógnitas  $w_i$  e  $m$  incógnitas  $P_i$ :

$$\sum_{i=1}^m w_i P_i^j = \begin{cases} \frac{2}{j+1}, & \text{se } j \text{ for par} \\ 0, & \text{se } j \text{ for ímpar} \end{cases}$$

$(j=0, \dots, m-1)$

→ A quadratura de Gauss fornece a solução exata, se para integrar um polinômio de ordem  $m$  forem utilizados  $\frac{m+1}{2}$  pontos de Gauss.  
( $m$  ímpar)

→ Com  $n$  pontos de Gauss integra-se de um modo exato um polinômio de grau  $2n-1$

Número de pontos de Gauss	Grav do polinômio	$P_i, W_i$
$M$	$2M-1$	
①	1	$P_1 = 0$ $W_1 = 2$
②	3	$P_1 = -1/\sqrt{3}$ $P_2 = 1/\sqrt{3}$ $W_1 = 1$ $W_2 = 1$
③	5	$P_1 = -\sqrt{3}/\sqrt{5}$ $P_2 = 0$ $P_3 = \sqrt{3}/\sqrt{5}$ $W_1 = 5/9$ $W_2 = 8/9$ $W_3 = 5/9$

Integral duplo

$$\int_{-1}^{+1} \int_{-1}^{+1} f(x, y) dx dy = \int_{-1}^{+1} \left[ \int_{-1}^{+1} f(x, y) dx \right] dy =$$

$$= \int_{-1}^{+1} \left[ \sum_{i=1}^m w_i f(P_i, y) \right] dy =$$

$$= \int_{-1}^{+1} g(y) dy = \sum_{j=1}^m w_j g(P_j) =$$

$$= \sum_{j=1}^m w_j \left[ \sum_{i=1}^m w_i f(P_i, P_j) \right] = \sum_{i=1}^m \sum_{j=1}^m w_i w_j f(P_i, P_j)$$

Integral triplo

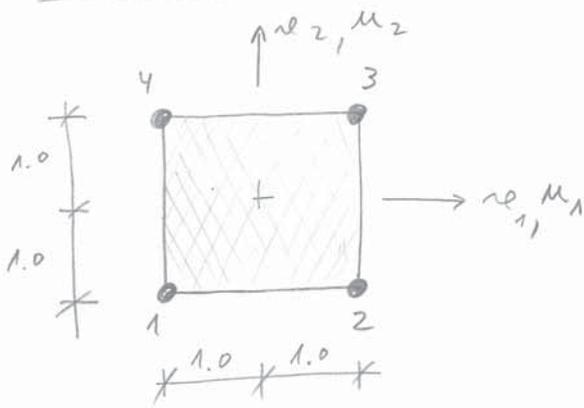
$$\int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} f(x, y, z) dx dy dz = \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^m w_i w_j w_k f(P_i, P_j, P_k)$$

PROBLEMA (4)

Calcular o integral da pag. MEF-17 recorrendo à quadratura de Gauss.

Fim da 6ª aula

7ª aula



$t \rightarrow$  espessura do elemento finito

MEF-24

$u_i \rightarrow$  campo de deslocamentos segundo  $r_i$

$a_{jk} \rightarrow$  deslocamento do nó  $j$  segundo  $r_k$

$N_j \rightarrow$  funções de forma associada ao nó  $j$

Os campos de deslocamentos  $u_1$  e  $u_2$  são interpolados separadamente. Considerando apenas  $u_1$ :

$$u_1(r_1, r_2) = N_1(r_1, r_2) a_{11} + N_2(r_1, r_2) a_{21} + N_3(r_1, r_2) a_{31} + N_4(r_1, r_2) a_{41}$$

$$u_1 = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} \Rightarrow u_i = \sum_{j=1}^4 N_j a_{ji}$$

$$u_2 = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ \text{---} \\ a_{21} \\ a_{22} \\ \text{---} \\ a_{31} \\ a_{32} \\ \text{---} \\ a_{41} \\ a_{42} \end{bmatrix} \begin{matrix} \left. \vphantom{\begin{matrix} a_{11} \\ a_{12} \end{matrix}} \right\} \text{Nó } 1 \\ \left. \vphantom{\begin{matrix} a_{21} \\ a_{22} \end{matrix}} \right\} \text{Nó } 2 \\ \left. \vphantom{\begin{matrix} a_{31} \\ a_{32} \end{matrix}} \right\} \text{Nó } 3 \\ \left. \vphantom{\begin{matrix} a_{41} \\ a_{42} \end{matrix}} \right\} \text{Nó } 4 \end{matrix}$$

$$u \sim N a$$

$$(2 \times 1) \quad (2 \times 8) \quad (8 \times 1)$$

$\rightarrow$  n. de graus de liberdade do elemento finito =  $4 \times 2$

n. de nós  $\leftarrow$   $\rightarrow$  n. de g.l. por nó

A função de forma  $N_j$  assume o valor unitário no nó  $j$  e é nula nos restantes nós.

$N_j$	$\eta_1$	$\eta_2$	$N_1$	$N_2$	$N_3$	$N_4$
1	-1	-1	1	0	0	0
2	1	-1	0	1	0	0
3	1	1	0	0	1	0
4	-1	1	0	0	0	1

$$N_1 = \frac{1}{4} (1 - \eta_1)(1 - \eta_2) = \frac{\eta_1 \eta_2}{4} - \frac{\eta_1}{4} - \frac{\eta_2}{4} + \frac{1}{4}$$

$$N_2 = \frac{1}{4} (1 + \eta_1)(1 - \eta_2) = -\frac{\eta_1 \eta_2}{4} + \frac{\eta_1}{4} - \frac{\eta_2}{4} + \frac{1}{4}$$

$$N_3 = \frac{1}{4} (1 + \eta_1)(1 + \eta_2) = \frac{\eta_1 \eta_2}{4} + \frac{\eta_1}{4} + \frac{\eta_2}{4} + \frac{1}{4}$$

$$N_4 = \frac{1}{4} (1 - \eta_1)(1 + \eta_2) = -\frac{\eta_1 \eta_2}{4} - \frac{\eta_1}{4} + \frac{\eta_2}{4} + \frac{1}{4}$$

Tratam-se de polinómios do 2º grau incompletos:

$$P(\eta_1, \eta_2) = a \eta_1^2 + b \eta_1 \eta_2 + c \eta_2^2 + d \eta_1 + e \eta_2 + f$$

estes termos  
não são  
considerados

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial u_1} & 0 \\ 0 & \frac{\partial}{\partial u_2} \\ \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Leftrightarrow \underset{(3 \times 1)}{\underline{\epsilon}} = \underset{(3 \times 2)}{\underline{L}} \underset{(2 \times 1)}{\underline{u}}$$

$$\underset{\sim}{\epsilon} = \underset{\sim}{L} \underset{\sim}{u}$$

$$\underset{\sim}{u} = \underset{\sim}{N} \underset{\sim}{a}$$

$$\underset{\sim}{\epsilon} = \underset{\sim}{L} \underset{\sim}{N} \underset{\sim}{a} \Leftrightarrow \underset{\sim}{\epsilon} = \underset{\sim}{B} \underset{\sim}{a} \rightsquigarrow \underset{\sim}{B} = \underset{\sim}{L} \underset{\sim}{N}$$

$(3 \times 1) \quad (3 \times 2)(2 \times 8)(8 \times 1) \qquad (3 \times 1) \quad (3 \times 8)(8 \times 1) \qquad (3 \times 8) \quad (3 \times 2)(2 \times 8)$

$$\underset{\sim}{B} = \begin{bmatrix} \frac{\partial N_1}{\partial u_1} & 0 & \frac{\partial N_2}{\partial u_1} & 0 & \frac{\partial N_3}{\partial u_1} & 0 & \frac{\partial N_4}{\partial u_1} & 0 \\ 0 & \frac{\partial N_1}{\partial u_2} & 0 & \frac{\partial N_2}{\partial u_2} & 0 & \frac{\partial N_3}{\partial u_2} & 0 & \frac{\partial N_4}{\partial u_2} \\ \frac{\partial N_1}{\partial u_2} & \frac{\partial N_1}{\partial u_1} & \frac{\partial N_2}{\partial u_2} & \frac{\partial N_2}{\partial u_1} & \frac{\partial N_3}{\partial u_2} & \frac{\partial N_3}{\partial u_1} & \frac{\partial N_4}{\partial u_2} & \frac{\partial N_4}{\partial u_1} \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} \frac{u_2}{4} - \frac{1}{4} & 0 & -\frac{u_2}{4} + \frac{1}{4} & 0 & \frac{u_2}{4} + \frac{1}{4} & 0 & -\frac{u_2}{4} - \frac{1}{4} & 0 \\ 0 & \frac{u_1}{4} - \frac{1}{4} & 0 & -\frac{u_1}{4} - \frac{1}{4} & 0 & \frac{u_1}{4} + \frac{1}{4} & 0 & -\frac{u_1}{4} + \frac{1}{4} \\ \frac{u_1}{4} - \frac{1}{4} & \frac{u_2}{4} - \frac{1}{4} & -\frac{u_1}{4} - \frac{1}{4} & -\frac{u_2}{4} + \frac{1}{4} & \frac{u_1}{4} + \frac{1}{4} & \frac{u_2}{4} + \frac{1}{4} & -\frac{u_1}{4} + \frac{1}{4} & -\frac{u_2}{4} - \frac{1}{4} \end{bmatrix}$$

# Estado Plano de Tensões (EPT)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 \\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} \Leftrightarrow \underset{(3 \times 1)}{\sigma} = \underset{(3 \times 3)}{D} \underset{(3 \times 1)}{\epsilon}$$

$$K = \int_V \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} dV \qquad K = \int_S \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} t dS$$

↙ espessura

$$K = \int_{-1}^{+1} \int_{-1}^{+1} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} t dr_1 dr_2$$

(8x8)                      (8x3) (3x3) (3x8)

Ex:

$$K_{ij} = \int_{-1}^{+1} \int_{-1}^{+1} \sum_{k=1}^3 \sum_{p=1}^3 \left( B_{ki} D_{kp} B_{pj} \right) t dr_1 dr_2$$

$$K_{11} = \int_{-1}^{+1} \int_{-1}^{+1} \sum_{k=1}^3 \sum_{p=1}^3 \left( B_{k1} D_{kp} B_{p1} \right) t dr_1 dr_2$$

$$\rightarrow \sum_{p=1}^3 \left( B_{11} D_{1p} B_{p1} \right) + \sum_{p=1}^3 \left( B_{21} D_{2p} B_{p1} \right) + \sum_{p=1}^3 \left( B_{31} D_{3p} B_{p1} \right) =$$

$$= B_{11} D_{11} B_{11} + B_{11} D_{12} B_{21} + B_{11} D_{13} B_{31} + B_{31} D_{31} B_{11} + B_{31} D_{32} B_{21} + B_{31} D_{33} B_{31} =$$

$$= B_{11} D_{11} B_{11} + B_{31} D_{33} B_{31} = B_{11}^2 D_{11} + B_{31}^2 D_{33}$$

$$K_{111} = \int_{-1}^{+1} \int_{-1}^{+1} \left[ \left( \frac{v_2}{4} - \frac{1}{4} \right)^2 \frac{E}{1-\nu^2} + \left( \frac{v_1}{4} - \frac{1}{4} \right)^2 \frac{E}{2(1+\nu)} \right] t \, dv_1 \, dv_2 =$$

$$= \frac{E t}{16} \int_{-1}^{+1} \int_{-1}^{+1} \left( \frac{v_2^2 - 2v_2 + 1}{1-\nu^2} + \frac{v_1^2 - 2v_1 + 1}{2(1+\nu)} \right) dv_1 \, dv_2$$

Supondo  $E=16$ ;  $t=1$ ;  $\nu=0$  vem:

$$K_{111} = \int_{-1}^{+1} \int_{-1}^{+1} \left( v_2^2 - 2v_2 + 1 + \frac{v_1^2}{2} - v_1 + \frac{1}{2} \right) dv_1 \, dv_2 =$$

$$= \int_{-1}^{+1} \int_{-1}^{+1} \left( \frac{v_1^2}{2} - v_1 + v_2^2 - 2v_2 + \frac{3}{2} \right) dv_1 \, dv_2 =$$

$$= \int_{-1}^{+1} \left[ \frac{v_1^3}{6} - \frac{v_1^2}{2} + v_1 v_2^2 - 2v_1 v_2 + \frac{3}{2} v_1 \right]_{-1}^{+1} dv_2 =$$

$$= \int_{-1}^{+1} \left( \frac{1}{6} + v_2^2 - 2v_2 + \frac{3}{2} \right) \times 2 \, dv_2 = \int_{-1}^{+1} \left( 2v_2^2 - 4v_2 + \frac{10}{3} \right) dv_2 =$$

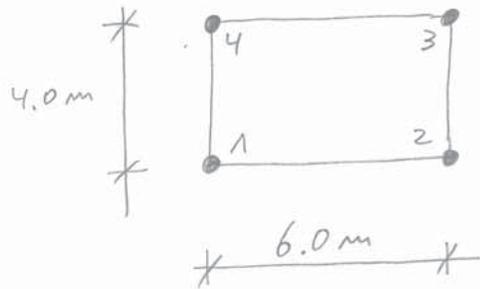
$$= \left[ \frac{2v_2^3}{3} - 2v_2^2 + \frac{10}{3} v_2 \right]_{-1}^{+1} = \left( \frac{2}{3} + \frac{10}{3} \right) \times 2 = 8.000.000$$

→ Confirmado  
pelo FEMIX

PROBLEMA 5

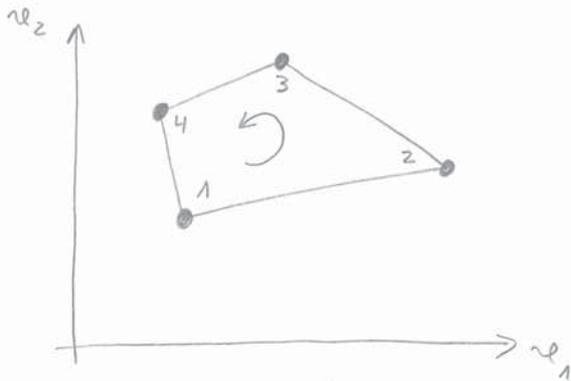
MEF-29

Calcular o elemento  $K_{58}$  da matriz de rigidez do seguinte elemento finito:

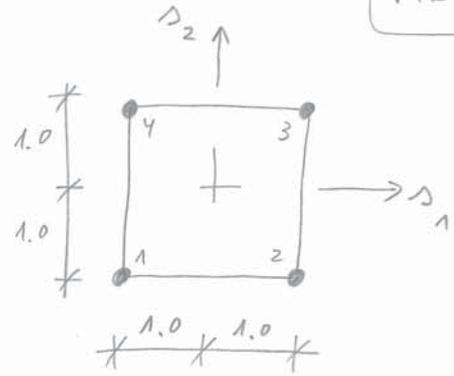


$$\begin{cases} E = 200\,000 \text{ MPa} \\ \nu = 0.3 \\ t = 0.2 \text{ m (constante)} \end{cases}$$

Fim da 7.<sup>a</sup> aula



Difícil de integrar  
no referencial  $(r_1, r_2)$



Transformação para  
o referencial  $(s_1, s_2)$

Considerando a função de forma  $N_j(r_1(s_1, s_2), r_2(s_1, s_2))$

$$\begin{cases} \frac{\partial N_j}{\partial s_1} = \frac{\partial N_j}{\partial r_1} \frac{\partial r_1}{\partial s_1} + \frac{\partial N_j}{\partial r_2} \frac{\partial r_2}{\partial s_1} \\ \frac{\partial N_j}{\partial s_2} = \frac{\partial N_j}{\partial r_1} \frac{\partial r_1}{\partial s_2} + \frac{\partial N_j}{\partial r_2} \frac{\partial r_2}{\partial s_2} \end{cases}$$

$$\begin{bmatrix} \frac{\partial N_j}{\partial s_1} \\ \frac{\partial N_j}{\partial s_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial s_1} & \frac{\partial r_2}{\partial s_1} \\ \frac{\partial r_1}{\partial s_2} & \frac{\partial r_2}{\partial s_2} \end{bmatrix} \begin{bmatrix} \frac{\partial N_j}{\partial r_1} \\ \frac{\partial N_j}{\partial r_2} \end{bmatrix} \Leftrightarrow \frac{\partial N_j}{\partial s_i} = \sum_{k=1}^2 \left( \frac{\partial N_j}{\partial r_k} \frac{\partial r_k}{\partial s_i} \right)$$

Matriz Jacobiana  $J \rightarrow J_{ik} = \frac{\partial r_k}{\partial s_i}$

$$\frac{\partial N_j}{\partial s_i} = \sum_{k=1}^2 \left( J_{ik} \frac{\partial N_j}{\partial r_k} \right)$$

$$\frac{\partial N_{ij}}{\partial \tilde{\rho}} = J \frac{\partial N_{ij}}{\partial \tilde{\rho}} \Leftrightarrow \frac{\partial N_{ij}}{\partial \tilde{\rho}} = J^{-1} \frac{\partial N_{ij}}{\partial \tilde{\rho}}$$

$(2 \times 1) \quad (2 \times 2) \quad (2 \times 1)$

$(2 \times 1) \quad (2 \times 2) \quad (2 \times 1)$

↪ elementos da matriz B

De acordo com MEF-25:

$$\left[ \begin{array}{l} N_1 = \frac{1}{4} (1 - \rho_1) (1 - \rho_2) \\ N_2 = \frac{1}{4} (1 + \rho_1) (1 - \rho_2) \\ N_3 = \frac{1}{4} (1 + \rho_1) (1 + \rho_2) \\ N_4 = \frac{1}{4} (1 - \rho_1) (1 + \rho_2) \end{array} \right. \left. \begin{array}{l} \frac{\partial N_1}{\partial \rho_1} = -\frac{1}{4} (1 - \rho_2) \\ \frac{\partial N_1}{\partial \rho_2} = -\frac{1}{4} (1 - \rho_1) \\ \vdots \\ \frac{\partial N_4}{\partial \rho_1} = -\frac{1}{4} (1 + \rho_2) \\ \frac{\partial N_4}{\partial \rho_2} = \frac{1}{4} (1 - \rho_1) \end{array} \right\} \frac{\partial N_j}{\partial \rho_i}$$

(j = 1, ..., 4)  
(i = 1, ..., 2)

Elementos da matriz jacobiana:  $J_{ik} = \frac{\partial \tilde{\rho}_k}{\partial \rho_i}$

$$\tilde{\rho}_k = N_1 \tilde{\rho}_{1k} + N_2 \tilde{\rho}_{2k} + N_3 \tilde{\rho}_{3k} + N_4 \tilde{\rho}_{4k} = \sum_{j=1}^4 (N_j \tilde{\rho}_{jk})$$

$$J_{ik} = \frac{\partial \tilde{\rho}_k}{\partial \rho_i} = \frac{\partial}{\partial \rho_i} \left[ \sum_{j=1}^4 (N_j \tilde{\rho}_{jk}) \right] =$$

$$= \frac{\partial N_1}{\partial \rho_i} \tilde{\rho}_{1k} + \frac{\partial N_2}{\partial \rho_i} \tilde{\rho}_{2k} + \frac{\partial N_3}{\partial \rho_i} \tilde{\rho}_{3k} + \frac{\partial N_4}{\partial \rho_i} \tilde{\rho}_{4k}$$

$$J_{ik} = \frac{\partial \tilde{\rho}_k}{\partial \rho_i} = \sum_{j=1}^4 \left( \frac{\partial N_j}{\partial \rho_i} \tilde{\rho}_{jk} \right)$$

$\updownarrow$   
 $c_{ik} = \sum_{j=1}^4 a_{ij} b_{jk} \Leftrightarrow \begin{matrix} c & = & a & b \\ (2 \times 2) & & (2 \times 4) & (4 \times 2) \end{matrix}$

↪ Elemento isoparamétrico

$N_j$  → função de forma associada ao nó j

$\tilde{\rho}_{jk}$  → coordenada do nó j, segundo  $\tilde{\rho}_k$

$$\tilde{K} = \int_V (\tilde{B}^T \tilde{D} \tilde{B}) dV = \int_S (\tilde{B}^T \tilde{D} \tilde{B}) t dS =$$

$\hookrightarrow$  área do elemento finito plano

$$= \int_S (\tilde{B}^T \tilde{D} \tilde{B}) t dr_1 dr_2 = \int_{-1}^{+1} \int_{-1}^{+1} (\tilde{B}^T \tilde{D} \tilde{B}) t |J| ds_1 ds_2$$

$\hookrightarrow$  determinante da matriz jacobiana (determinante jacobiano)

Algoritmo para o cálculo de  $\tilde{K}$  recorrendo

à quadratura de Gauss:

$$\tilde{K} = \sum_{i=1}^m \sum_{j=1}^m w_i w_j \left( \tilde{B}^T \tilde{D} \tilde{B} t |J| \right)_{(P_i, P_j)}$$

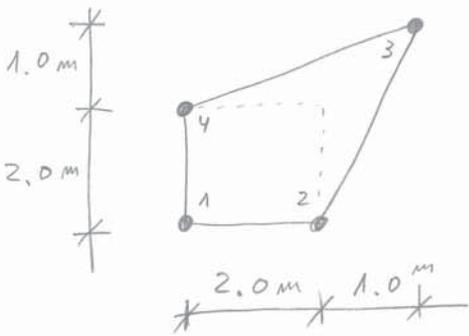
$\rightarrow$  Para cada ponto de Gauss  $(P_i, P_j)$ :

- calcular  $\frac{\partial N_j}{\partial s_i}$  no ponto  $(P_i, P_j)$
- calcular a matriz jacobiana:  $J_{iK} = \frac{\partial r_K}{\partial s_i} = \sum_{j=1}^4 \left( \frac{\partial N_j}{\partial s_i} \tilde{r}_{e_j K} \right)$
- calcular a inversa da matriz jacobiana  $J^{-1}$
- calcular o determinante jacobiano  $|J|$
- calcular a espessura  $t$  em  $(P_i, P_j)$
- calcular a matriz  $\tilde{B}$   $\implies \frac{\partial N_j}{\partial r} = J^{-1} \frac{\partial N_j}{\partial s}$   
(3x8)
- calcular a matriz  $\tilde{D}$  (3x3)
- calcular  $w_i w_j \left( \tilde{B}^T \tilde{D} \tilde{B} t |J| \right)_{(P_i, P_j)}$
- acumular em  $\tilde{K}$

# PROBLEMA (6)

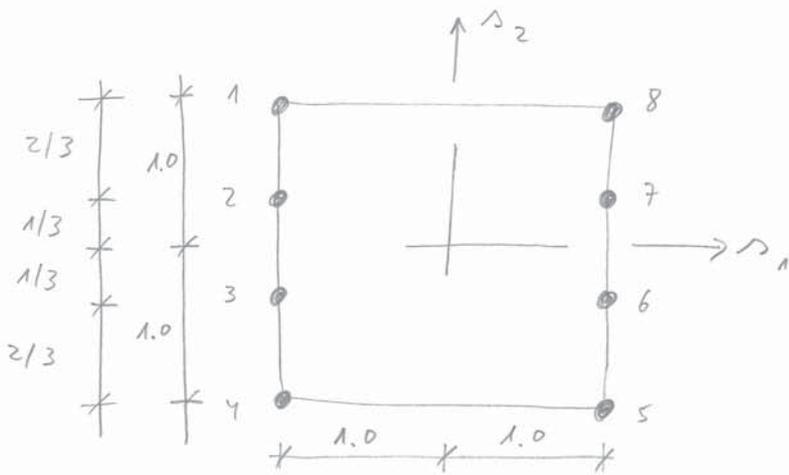
MEF-33

Calcular o elemento  $K_{58}$  da matriz de rigidez do seguinte elemento finito:



$$\begin{cases} E = 200\,000 \text{ MPa} \\ \nu = 0.3 \\ t = 0.2 \text{ m (constante)} \end{cases}$$

Fim da 8.ª aula



$$N_j \rightarrow \begin{cases} = 1 \text{ no nó } j \\ = 0 \text{ nos restantes nós} \end{cases}$$

$m \rightarrow$  n.º de nós

$N_j \rightarrow$  função de forma associada ao nó  $j$  ( $j = 1, \dots, m$ )

$m = 8$  (neste exemplo)

$\bar{x}_{jk}$   $\rightarrow$  coordenada do nó  $j$  segundo  $\bar{x}_k$

$a_j \rightarrow$  deslocamento  $u$  no nó  $j$

Interpolação do campo escalar  $u(\bar{x}_1, \bar{x}_2)$ :

$$u(\bar{x}_1, \bar{x}_2) = \alpha_1 + \alpha_2 \bar{x}_1 + \alpha_3 \bar{x}_2 + \alpha_4 \bar{x}_1 \bar{x}_2 + \alpha_5 \bar{x}_2^2 + \alpha_6 \bar{x}_1 \bar{x}_2^2 + \alpha_7 \bar{x}_2^3 + \alpha_8 \bar{x}_1 \bar{x}_2^3$$

$$u(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} 1 & \bar{x}_1 & \bar{x}_2 & \bar{x}_1 \bar{x}_2 & \bar{x}_2^2 & \bar{x}_1 \bar{x}_2^2 & \bar{x}_2^3 & \bar{x}_1 \bar{x}_2^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}$$

variáveis dos termos do polinómio

Coefficientes dos termos do polinómio

$$u(\bar{x}_1, \bar{x}_2) = \underset{(1 \times 1)}{P}(\bar{x}_1, \bar{x}_2) \underset{(1 \times m)}{\alpha}$$

$m = 8$  (número de nós) (número de termos)

$$u = P \alpha$$

Atribuindo a  $\bar{x}_1$  e  $\bar{x}_2$  os valores das coordenadas dos diversos nós

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} 1 & \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{11} \bar{x}_{12} & \bar{x}_{12}^2 & \bar{x}_{11} \bar{x}_{12}^2 & \bar{x}_{12}^3 & \bar{x}_{11} \bar{x}_{12}^3 \\ 1 & \bar{x}_{21} & \bar{x}_{22} & \bar{x}_{21} \bar{x}_{22} & \bar{x}_{22}^2 & \bar{x}_{21} \bar{x}_{22}^2 & \bar{x}_{22}^3 & \bar{x}_{21} \bar{x}_{22}^3 \\ 1 & \bar{x}_{31} & \bar{x}_{32} & \bar{x}_{31} \bar{x}_{32} & \bar{x}_{32}^2 & \bar{x}_{31} \bar{x}_{32}^2 & \bar{x}_{32}^3 & \bar{x}_{31} \bar{x}_{32}^3 \\ 1 & \bar{x}_{41} & \bar{x}_{42} & \bar{x}_{41} \bar{x}_{42} & \bar{x}_{42}^2 & \bar{x}_{41} \bar{x}_{42}^2 & \bar{x}_{42}^3 & \bar{x}_{41} \bar{x}_{42}^3 \\ 1 & \bar{x}_{51} & \bar{x}_{52} & \bar{x}_{51} \bar{x}_{52} & \bar{x}_{52}^2 & \bar{x}_{51} \bar{x}_{52}^2 & \bar{x}_{52}^3 & \bar{x}_{51} \bar{x}_{52}^3 \\ 1 & \bar{x}_{61} & \bar{x}_{62} & \bar{x}_{61} \bar{x}_{62} & \bar{x}_{62}^2 & \bar{x}_{61} \bar{x}_{62}^2 & \bar{x}_{62}^3 & \bar{x}_{61} \bar{x}_{62}^3 \\ 1 & \bar{x}_{71} & \bar{x}_{72} & \bar{x}_{71} \bar{x}_{72} & \bar{x}_{72}^2 & \bar{x}_{71} \bar{x}_{72}^2 & \bar{x}_{72}^3 & \bar{x}_{71} \bar{x}_{72}^3 \\ 1 & \bar{x}_{81} & \bar{x}_{82} & \bar{x}_{81} \bar{x}_{82} & \bar{x}_{82}^2 & \bar{x}_{81} \bar{x}_{82}^2 & \bar{x}_{82}^3 & \bar{x}_{81} \bar{x}_{82}^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}$$

$$\underset{\sim}{a} = \underset{\sim}{C} \underset{\sim}{x}$$

(8x1)    (8x8) (8x1)

$$\underset{\sim}{x} = \underset{\sim}{C}^{-1} \underset{\sim}{a}$$

(8x1)    (8x8) (8x1)

$N_{\sigma_j}$	$\bar{\sigma}_{j1}$	$\bar{\sigma}_{j2}$
1	-1	1
2	-1	1/3
3	-1	-1/3
4	-1	-1
5	1	-1
6	1	-1/3
7	1	1/3
8	1	1

$$\underset{\sim}{u} = \underset{\sim}{P} \underset{\sim}{x}$$

↑

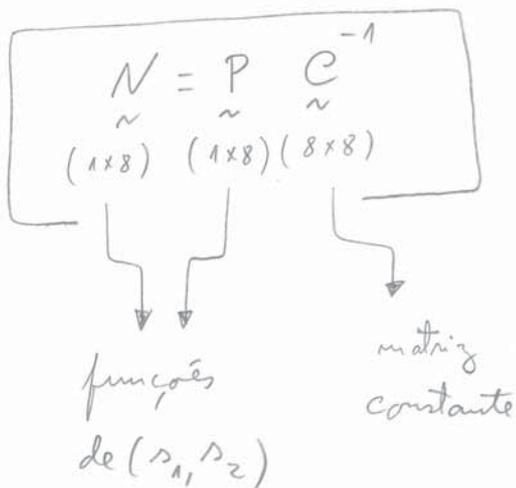
$$\underset{\sim}{x} = \underset{\sim}{C}^{-1} \underset{\sim}{a}$$

$$\underset{\sim}{u} = \underset{\sim}{P} \underset{\sim}{C}^{-1} \underset{\sim}{a}$$

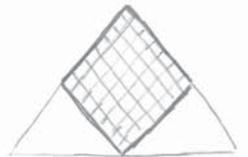
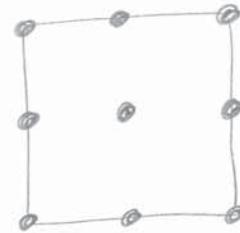
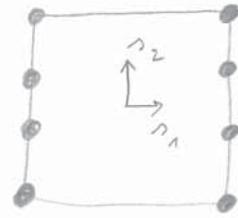
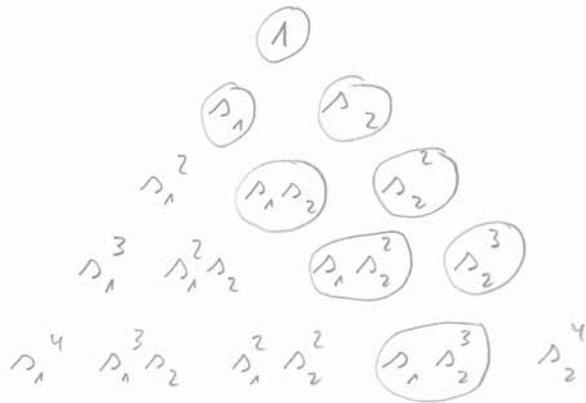
(1x1)    (1x8) (8x8) (8x1)

$$\underset{\sim}{u} = \underset{\sim}{N} \underset{\sim}{a}$$

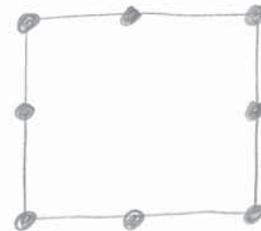
(1x1)    (1x8) (8x1)



# Triângulo de Pascal



Elemento Lagrangeano



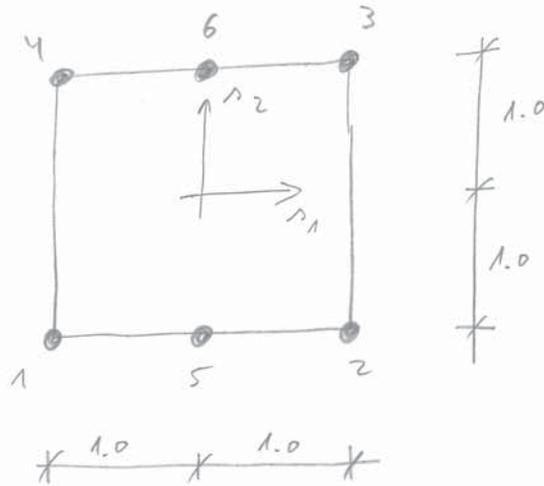
Elemento "Serendipity"

No caso tridimensional ( $\rho_1, \rho_2, \rho_3$ ), estes conceitos generalizam-se a uma pirâmide triangular de termos.

PROBLEMA (7)

MEF-37

Calcular a função de forma  $N_5$  do seguinte elemento finito



Fim da 9.<sup>a</sup> aula

PROBLEMA 8

1.6m

$p = 15 \text{ kN/m}$

$t = 0.35 \text{ m}$

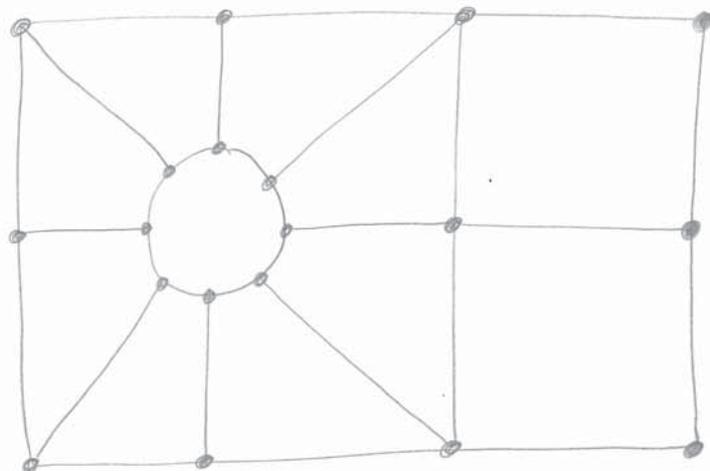
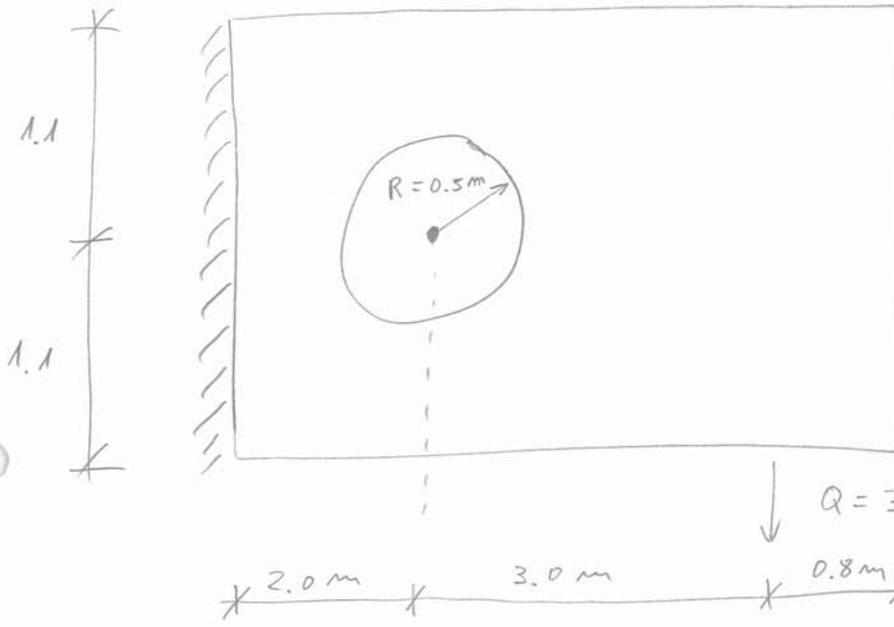
$E = 25\,000 \text{ MPa}$

$\nu = 0.15$

$\rho = 2500 \text{ Kg/m}^3$

$g = -10 \text{ m/s}^2$

$g = -10 \text{ m/s}^2$



Elementos de  $\begin{cases} 4 \text{ nós} \\ 8 \text{ nós} \\ 9 \text{ nós} \end{cases}$

Refinamento  $\begin{cases} \text{baixo} \\ \text{elevado} \end{cases}$

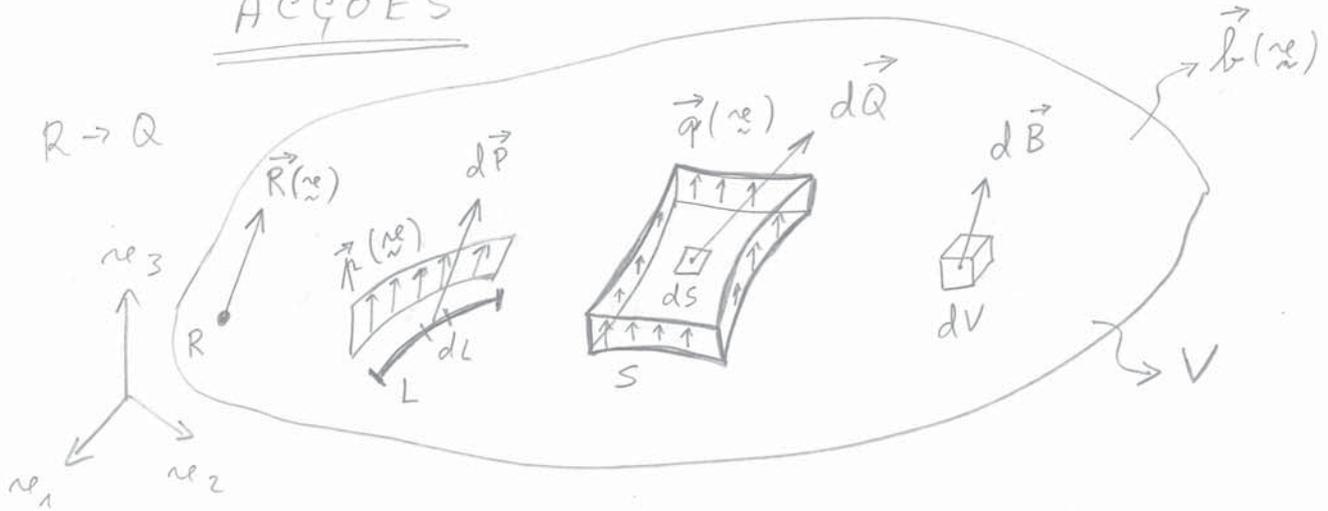
N. de pontos de Gauss  $\begin{cases} 2 \\ 3 \end{cases}$

(12 casos)

↳ macro elementos

Fim da 10ª aula

AÇÕES



- $\vec{R} \rightarrow$  ação concentrada no ponto  $R$
- $\vec{p} \rightarrow$  ação distribuída por unidade de comprimento no arco  $L$
- $\vec{q} \rightarrow$  ação distribuída por unidade de área em  $S$
- $\vec{b} \rightarrow$  ação distribuída por unidade de volume em  $V$

$$\begin{cases} d\vec{P} = \vec{p}(\underline{x}) dL \\ d\vec{Q} = \vec{q}(\underline{x}) dS \\ d\vec{B} = \vec{b}(\underline{x}) dV \end{cases}$$

$$\vec{b}(\underline{x}) = \rho(\underline{x}) \vec{g}$$

- $\rightarrow$  aceleração (afeta todo o domínio)
- $\rightarrow$  massa específica

trabalho interno = trabalho externo

$$\int_V \underline{\underline{\epsilon}}^T \underline{\underline{\sigma}} dV = \sum_R \underline{\underline{u}}^T \underline{\underline{R}} + \sum_L \int \underline{\underline{u}}^T \underline{\underline{p}} dL + \sum_S \int \underline{\underline{u}}^T \underline{\underline{q}} dS + \sum_V \int \underline{\underline{u}}^T \underline{\underline{b}} dV$$

→  $\underline{\underline{R}}(\underline{\underline{u}})$  é nulo em todos os pontos do domínio, excetuando o ponto de aplicação de cada ação concentrada

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

$$\underline{\underline{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\underline{\underline{R}} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

$$\underline{\underline{p}} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\underline{\underline{q}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\underline{\underline{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\left. \begin{array}{l} \underline{\underline{\sigma}} = \underline{\underline{D}} \underline{\underline{\epsilon}} \\ \underline{\underline{\epsilon}} = \underline{\underline{B}} \underline{\underline{a}} \end{array} \right\} \underline{\underline{\sigma}} = \underline{\underline{D}} \underline{\underline{B}} \underline{\underline{a}}$$

$m \rightarrow m$ : de nós do elemento

$$\underline{\underline{\epsilon}}^T \underline{\underline{\sigma}} = \underline{\underline{a}}^T \underline{\underline{B}}^T \underline{\underline{D}} \underline{\underline{B}} \underline{\underline{a}}$$

$$\underline{\underline{u}} = \underline{\underline{N}} \underline{\underline{a}} \Rightarrow \underline{\underline{u}}^T = \underline{\underline{a}}^T \underline{\underline{N}}^T$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (3 \times 1)$$

$$= \begin{bmatrix} N_1 & 0 & 0 & | & N_2 & 0 & 0 & | & \dots & | & N_m & 0 & 0 \\ 0 & N_1 & 0 & | & 0 & N_2 & 0 & | & \dots & | & 0 & N_m & 0 \\ 0 & 0 & N_1 & | & 0 & 0 & N_2 & | & \dots & | & 0 & 0 & N_m \end{bmatrix} \quad (3 \times 3m)$$

$$\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ \dots \\ a_{21} \\ a_{22} \\ a_{23} \\ \dots \\ \vdots \\ a_{m1} \\ a_{m2} \\ a_{m3} \end{bmatrix} \quad (3m \times 1)$$

$$\int_V \tilde{a}^T \tilde{B}^T \tilde{D} \tilde{B} \tilde{a} dV =$$

$$= \sum_R \tilde{a}^T \tilde{N}^T \tilde{R} + \sum_P \int_L \tilde{a}^T \tilde{N}^T \tilde{p} dL + \sum_Q \int_S \tilde{a}^T \tilde{N}^T \tilde{q} dS + \sum_b \int_V \tilde{a}^T \tilde{N}^T \tilde{b} dV$$

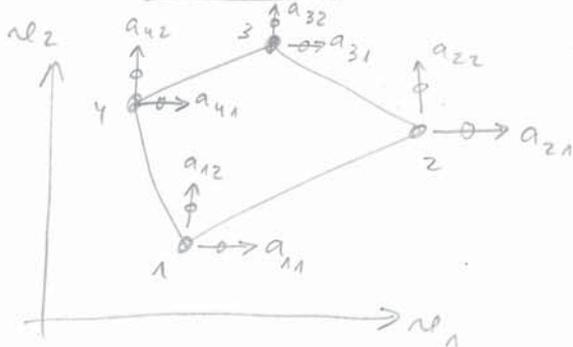
$$\left[ \int_V \tilde{B}^T \tilde{D} \tilde{B} dV \right] \tilde{a} = \sum_R \tilde{N}^T \tilde{R} + \sum_P \int_L \tilde{N}^T \tilde{p} dL + \sum_Q \int_S \tilde{N}^T \tilde{q} dS + \sum_b \int_V \tilde{N}^T \tilde{b} dV$$

$$K \tilde{a} = \sum_R \tilde{F}^{(R)} + \sum_P \tilde{F}^{(P)} + \sum_Q \tilde{F}^{(Q)} + \sum_b \tilde{F}^{(b)}$$

$$\boxed{K \tilde{a} = \tilde{F}}$$

### Estado Plano de Tensões

#### Elemento Finito de 4 nós



$$\tilde{a} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ a_{31} \\ a_{32} \\ a_{41} \\ a_{42} \end{bmatrix}; \tilde{F} = \begin{bmatrix} F_{11} \\ F_{12} \\ F_{21} \\ F_{22} \\ F_{31} \\ F_{32} \\ F_{41} \\ F_{42} \end{bmatrix}$$

$$K \tilde{a} = \tilde{F}$$

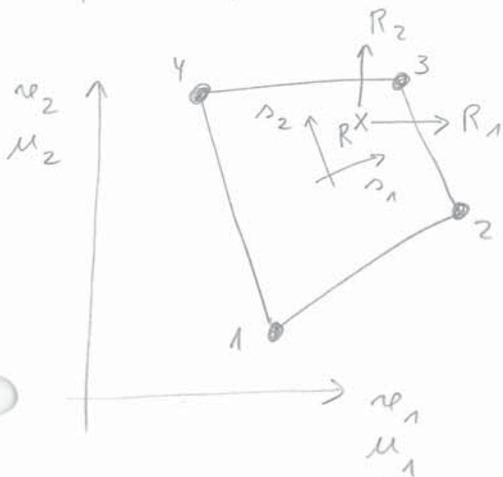
$(8 \times 8) (8 \times 1) (8 \times 1)$

$$\boxed{m = 4}$$

$$\boxed{2m = 8}$$

Elemento Finito de 4 nós

Considerando uma só força  $\vec{R}$  a actuar num ponto qualquer de um elemento finito:



O ponto R é definido com as coordenadas locais  $(\bar{s}_{R1}, \bar{s}_{R2})$

$$\underset{(8 \times 1)}{\tilde{F}}^{(R)} = \underset{(8 \times 2)}{N^T} \underset{(2 \times 1)}{R}$$

valores das funções de forma no ponto  $(\bar{s}_{R1}, \bar{s}_{R2})$

$$\begin{bmatrix} F_{11} \\ F_{12} \\ \dots \\ F_{21} \\ F_{22} \\ \dots \\ F_{31} \\ F_{32} \\ \dots \\ F_{41} \\ F_{42} \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ \dots \\ N_2 & 0 \\ 0 & N_2 \\ \dots \\ N_3 & 0 \\ 0 & N_3 \\ \dots \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

$(8 \times 1)$        $(8 \times 2)$        $(2 \times 1)$

$F_{jk} \rightarrow$  força no nó  $j$  segundo  $e_k$   
 $R_i \rightarrow$  acção concentrada segundo  $e_i$ .

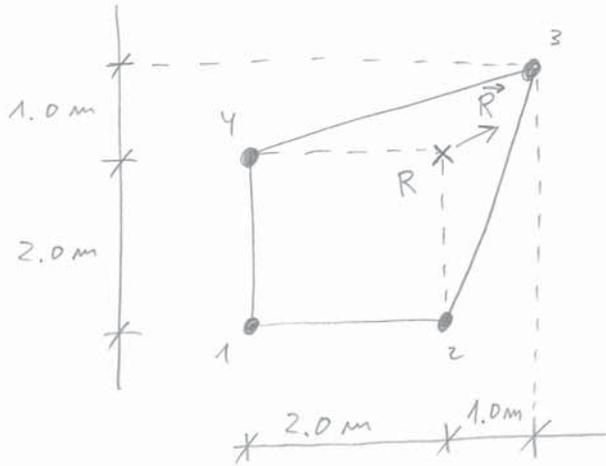
Se a acção concentrada actuar, por exemplo, no nó 2:

$$\begin{cases} \bar{s}_{R1} = 1 \\ \bar{s}_{R2} = -1 \end{cases} \begin{cases} N_1(1, -1) = 0 \\ N_2(1, -1) = 1 \\ N_3(1, -1) = 0 \\ N_4(1, -1) = 0 \end{cases} \quad \underset{\sim}{F}^{(R)} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ R_1 \\ R_2 \\ \dots \\ 0 \\ 0 \end{bmatrix}$$

PROBLEMA (9)

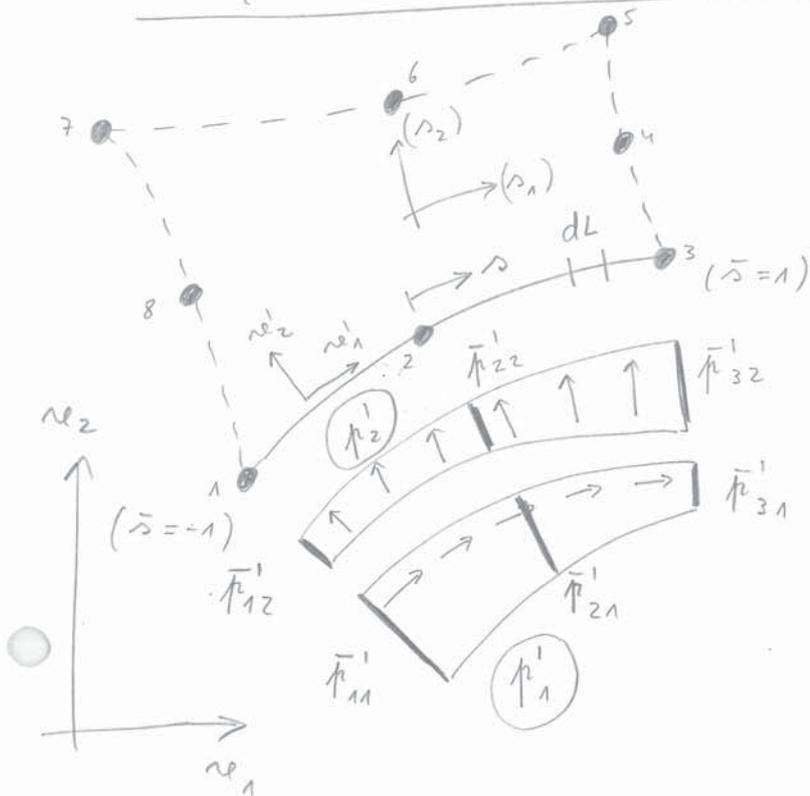
MEF-43

Calcular as forças nodais equivalentes à seguinte ação (ver o problema (6)):



$$\vec{R} = (100, 150) \text{ (kN)}$$

Fim da 11ª aula



$\bar{p}$  → valor modal

$p^i$  → ações distribuídas (no referencial local,

$\bar{p}_{jK}^{-1}$  → ações distribuídas por unidade de comprimento de arco, no nó  $j$ , segundo  $re'_K$

$p^i_1$  → positivo quando segue a numeração local dos nós ( $1 \rightarrow 2 \rightarrow 3$ )

$p^i_2$  → forma um referencial directo com  $p^i_1$

● Funções de forma no arco:

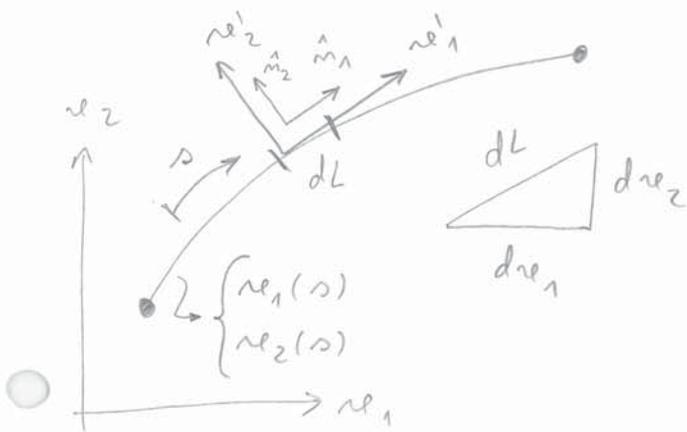
$$\begin{cases} \bar{N}_1(s) = \frac{1}{2}s^2 - \frac{1}{2}s \\ \bar{N}_2(s) = -s^2 + 1 \\ \bar{N}_3(s) = \frac{1}{2}s^2 + \frac{1}{2}s \end{cases}$$

$$p^i_1(s) = \bar{N}_1(s) \bar{p}^i_{11} + \bar{N}_2(s) \bar{p}^i_{21} + \bar{N}_3(s) \bar{p}^i_{31}$$

$$p^i_2(s) = \bar{N}_1(s) \bar{p}^i_{12} + \bar{N}_2(s) \bar{p}^i_{22} + \bar{N}_3(s) \bar{p}^i_{32}$$

$$p^i_K = \sum_{j=1}^3 \bar{N}_j \bar{p}^i_{jK}$$

$$\begin{cases} dP'_1 = p'_1 dL \\ dP'_2 = p'_2 dL \end{cases} \Rightarrow \begin{matrix} dP' = p' dL \\ (2 \times 1) \quad (2 \times 1) \quad (1 \times 1) \end{matrix}$$



$$\begin{cases} \hat{m}_1 \rightarrow \text{versor de } r'_1 \\ \hat{m}_2 \rightarrow \text{versor de } r'_2 \end{cases}$$

$$\vec{m}_1 = \left( \frac{dr_1}{ds}, \frac{dr_2}{ds} \right)$$

↳ vector tangente ao arco

$$\|\vec{m}_1\| = \sqrt{\left(\frac{dr_1}{ds}\right)^2 + \left(\frac{dr_2}{ds}\right)^2}$$

O arco é definido pelas

$$\text{funções } \begin{cases} r_1(s) \\ r_2(s) \end{cases}$$

$$\hat{m}_1 = \frac{\vec{m}_1}{\|\vec{m}_1\|} = (t_1, t_2)$$

$$\hat{m}_2 = (-t_2, t_1)$$

$$\underset{\sim}{r}' = \underset{\sim}{T} \underset{\sim}{r}$$

↳ matriz de transformação do referencial geral para o local

$$\begin{bmatrix} r'_1 \\ r'_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \\ -t_2 & t_1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Nota: as linhas de  $\underset{\sim}{T}$  são os versores dos eixos do referencial local

A transformação inversa é:

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} t_1 & -t_2 \\ t_2 & t_1 \end{bmatrix} \begin{bmatrix} r'_1 \\ r'_2 \end{bmatrix} \Rightarrow \underset{\sim}{r} = \underset{\sim}{T}^T \underset{\sim}{r}'$$

$$\hat{\vec{m}}_1 = \frac{1}{\|\vec{m}_1\|} \left( \frac{dr_{e1}}{ds}, \frac{dr_{e2}}{ds} \right) = (t_1, t_2)$$

$$t_1 = \frac{1}{\|\vec{m}_1\|} \frac{dr_{e1}}{ds} \quad ; \quad t_2 = \frac{1}{\|\vec{m}_1\|} \frac{dr_{e2}}{ds}$$

$$\begin{bmatrix} r_{e1} \\ r_{e2} \end{bmatrix} = \frac{1}{\|\vec{m}_1\|} \underbrace{\begin{bmatrix} \frac{dr_{e1}}{ds} & -\frac{dr_{e2}}{ds} \\ \frac{dr_{e2}}{ds} & \frac{dr_{e1}}{ds} \end{bmatrix}}_{\tilde{T}^T} \begin{bmatrix} r_{e1}' \\ r_{e2}' \end{bmatrix} \Rightarrow \begin{array}{c} r_{e} = \frac{1}{\|\vec{m}_1\|} \tilde{T}^T r_{e}' \\ \sim \quad \sim \quad \sim \end{array}$$

$$\tilde{T} = \frac{1}{\|\vec{m}_1\|} \tilde{T}$$

Tal como nos elementos isoparamétricos:

$$\begin{cases} r_{e1}(s) = \bar{N}_1(s) \bar{r}_{e11} + \bar{N}_2(s) \bar{r}_{e21} + \bar{N}_3(s) \bar{r}_{e31} \\ r_{e2}(s) = \bar{N}_1(s) \bar{r}_{e12} + \bar{N}_2(s) \bar{r}_{e22} + \bar{N}_3(s) \bar{r}_{e32} \end{cases} \quad \left. r_{eK} = \sum_{j=1}^3 \left( \bar{N}_j \bar{r}_{jK} \right) \right\}$$

→ m: de nós no arco

$$\frac{dr_{eK}}{ds} = \sum_{j=1}^3 \left( \frac{d\bar{N}_j}{ds} \bar{r}_{jK} \right)$$

$\bar{N}_j$  → função de forma (arco) associada ao nó  $j$

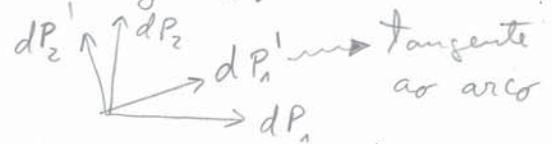
$\bar{r}_{jK}$  → coordenada do nó  $j$ , segundo  $r_{eK}$

Passagem de  $dP'$  para o referencial geral:

$$dP_{\sim} = \tilde{T}^T dP'$$

$$p_{\sim} dL = \tilde{T}^T p' dL$$

$$p_{\sim} = \tilde{T}^T p'$$





$$dL = \sqrt{dr_{e_1}^2 + dr_{e_2}^2}$$

$$\begin{cases} r_{e_1}(s) \rightsquigarrow dr_{e_1} = \frac{dr_{e_1}}{ds} ds \\ r_{e_2}(s) \rightsquigarrow dr_{e_2} = \frac{dr_{e_2}}{ds} ds \end{cases}$$

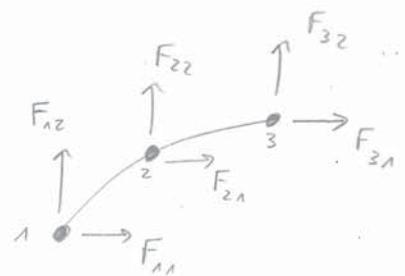
Nota:  $r_{e_1}$  e  $r_{e_2}$  s3o dependem de  $s$

$$dL = \sqrt{\left(\frac{dr_{e_1}}{ds} ds\right)^2 + \left(\frac{dr_{e_2}}{ds} ds\right)^2} = \sqrt{\left(\frac{dr_{e_1}}{ds}\right)^2 + \left(\frac{dr_{e_2}}{ds}\right)^2} ds$$

$$\frac{dL}{ds} = \sqrt{\left(\frac{dr_{e_1}}{ds}\right)^2 + \left(\frac{dr_{e_2}}{ds}\right)^2} \Rightarrow \frac{dL}{ds} = \|\vec{m}_1\|$$

$$\tilde{F}^{(r)} = \int_L \tilde{N}^T \tilde{p} dL = \int_{-1}^{+1} \tilde{N}^T \tilde{p} \frac{dL}{ds} ds = \int_{-1}^{+1} \tilde{N}^T \tilde{T}^T \tilde{p}' \|\vec{m}_1\| ds =$$

$$= \int_{-1}^{+1} \tilde{N}^T \frac{1}{\|\vec{m}_1\|} \tilde{T}^T \tilde{p}' \|\vec{m}_1\| ds$$



$$\tilde{F}^{(r)} = \int_{-1}^{+1} \tilde{N}^T \tilde{T}^T \tilde{p}' ds$$

(6x1)

↪ graus de liberdade dos nós do arco

$$\begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ \hline N_2 & 0 \\ 0 & N_2 \\ \hline N_3 & 0 \\ 0 & N_3 \end{bmatrix} = \begin{bmatrix} \bar{N}_1 & 0 \\ 0 & \bar{N}_1 \\ \hline \bar{N}_2 & 0 \\ 0 & \bar{N}_2 \\ \hline \bar{N}_3 & 0 \\ 0 & \bar{N}_3 \end{bmatrix}$$

$$N_{\bar{j}}(\bar{r}_1, -1) = \bar{N}_{\bar{j}}(\bar{s})$$

$$(\bar{j} = 1, \dots, 3)$$

$$\underset{\sim}{F}^{(\uparrow)} = \begin{bmatrix} F_{11} \\ F_{12} \\ \dots \\ F_{21} \\ F_{22} \\ \dots \\ F_{31} \\ F_{32} \end{bmatrix} = \int_{-1}^{+1} \begin{bmatrix} \bar{N}_1 & 0 \\ 0 & \bar{N}_1 \\ \dots \\ \bar{N}_2 & 0 \\ 0 & \bar{N}_2 \\ \dots \\ \bar{N}_3 & 0 \\ 0 & \bar{N}_3 \end{bmatrix} \begin{bmatrix} \frac{du_1}{ds} & -\frac{du_2}{ds} \\ \frac{du_2}{ds} & \frac{du_1}{ds} \end{bmatrix} \begin{bmatrix} p_1' \\ p_2' \end{bmatrix} ds$$

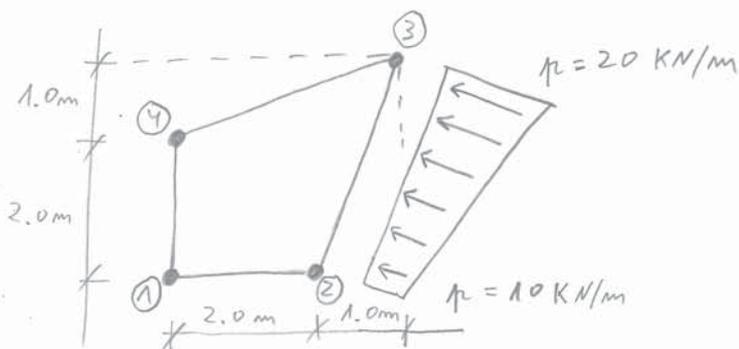
Este integral pode ser calculado recorrendo:

• à quadratura de Gauss.

As componentes de  $\underset{\sim}{F}^{(\uparrow)}$  relativas aos nós 4, ..., 8 são nulas.

### PROBLEMA (10)

Calcular as forças nodais equivalentes à seguinte ação:



Fim da 12ª aula

Forças de volume

$\vec{b}(\vec{x}) \rightarrow$  ação distribuída por unidade de volume

$dV$    $d\vec{B} = \vec{b} dV$

$$\vec{F}^{(b)} = \int_V \vec{N}^T \vec{b} dV$$

$$\vec{b} = \rho \vec{g}$$

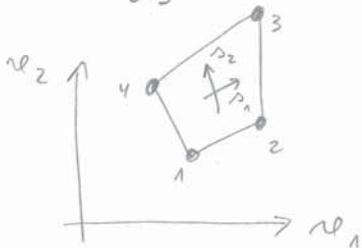
$$\vec{F}^{(b)} = \int_V \vec{N}^T \rho \vec{g} dV$$

$\rho \rightarrow$  massa específica  
(pode ser variável)  
(é um escalar)

$\vec{g} \rightarrow$  aceleração imposta  
(vem com três componentes)

Caso particular: estado plano de tensão

$$\vec{F}^{(b)} = \int_S \vec{N}^T \vec{g} \rho t ds$$



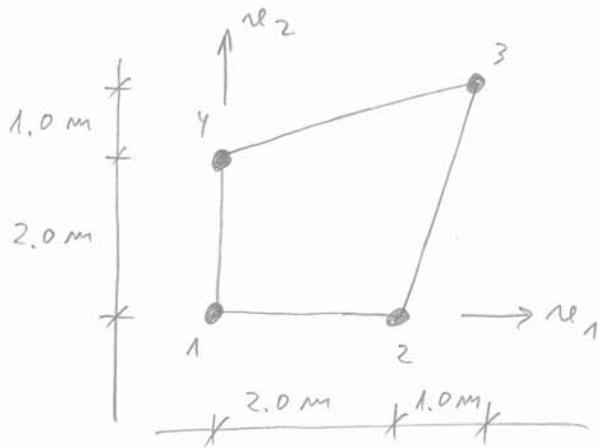
$t \rightarrow$  espessura  
(pode ser variável)

$$\vec{F}^{(b)}_{(8 \times 1)} = \int_{-1}^{+1} \int_{-1}^{+1} \vec{N}^T \vec{g} \rho t |\vec{J}| ds_1 ds_2$$

$(8 \times 2)$      $(2 \times 1)$

$\hookrightarrow$  calcular recorrendo à quadratura de Gauss

Calcular as forças nodais equivalentes à seguinte ação (ver o problema 6):



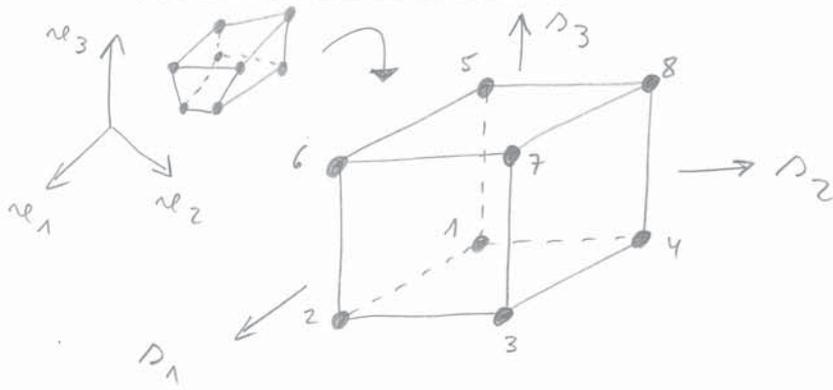
$$t = 0.2 \text{ m (constante)}$$

$$\rho = 2500 \text{ Kg/m}^3$$

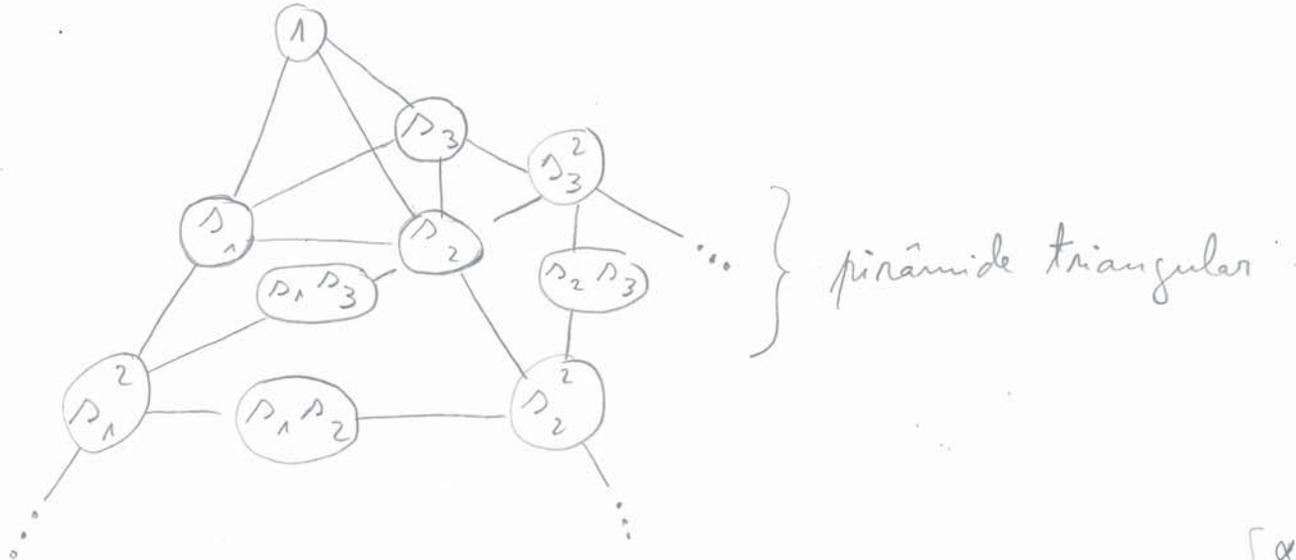
$$\vec{g} = (2, -10) \text{ (m/s}^2\text{)}$$

# Elementos tridimensionais (bricks)

MEF-51



$$\rho_1, \rho_2, \rho_3 \in [-1, +1]$$



pirâmide triangular

$$u(\rho_1, \rho_2, \rho_3) = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \rho_1 \rho_2 & \rho_2 \rho_3 & \rho_1 \rho_3 & \rho_1 \rho_2 \rho_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}$$

$$\left. \begin{aligned} u &= P \alpha \\ \sim & \sim \\ \alpha &= C^{-1} a \end{aligned} \right\} \begin{aligned} u &= P C^{-1} a \\ \sim & \sim \sim \\ N &= P C^{-1} \\ \sim & \sim \sim \end{aligned}$$

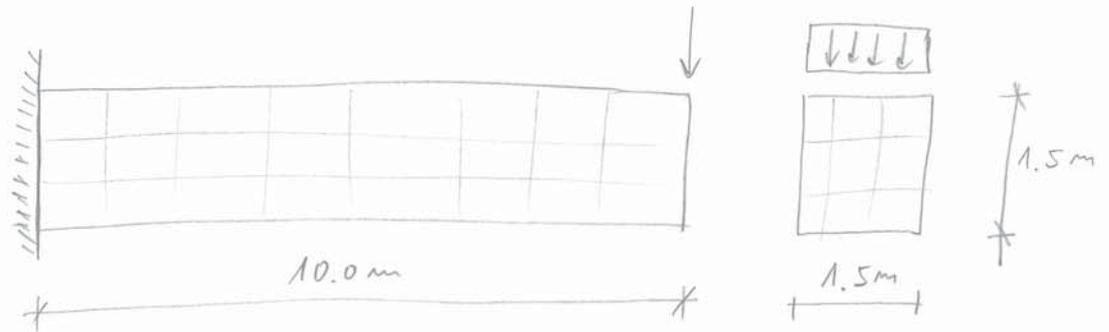


PROBLEMA

12

MEF-53

$$q = 1.3 \text{ MN/m}$$



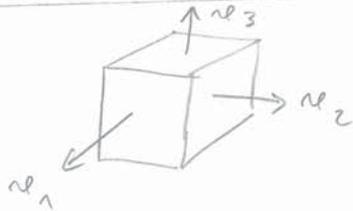
$$\begin{cases} E = 200\,000 \text{ MPa} \\ \nu = 0.3 \end{cases}$$

Calcular os deslocamentos e as tensões na consola recorrendo a elementos finitos tridimensionais.

Fim da 13ª aula

Relações tensão - deformaçãoIsotropia (3D)

$$\sigma_{ij} = \frac{E}{2(1+\nu)} \gamma_{ij} \Leftrightarrow \gamma_{ij} = \frac{2(1+\nu)}{E} \sigma_{ij}$$



$$\begin{cases} \epsilon_1 = \frac{1}{E} \sigma_1 - \frac{\nu}{E} \sigma_2 - \frac{\nu}{E} \sigma_3 \\ \epsilon_2 = -\frac{\nu}{E} \sigma_1 + \frac{1}{E} \sigma_2 - \frac{\nu}{E} \sigma_3 \\ \epsilon_3 = -\frac{\nu}{E} \sigma_1 - \frac{\nu}{E} \sigma_2 + \frac{1}{E} \sigma_3 \end{cases}$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} A & B & B \\ B & A & B \\ B & B & A \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \quad \left( A = \frac{1}{E} \right)$$

$$\left( B = -\frac{\nu}{E} \right)$$

Invertendo:  $\rightarrow \begin{bmatrix} \sigma \\ \sim \\ \sim \end{bmatrix} = \begin{bmatrix} D \\ \sim \\ \sim \end{bmatrix} \begin{bmatrix} \epsilon \\ \sim \\ \sim \end{bmatrix}$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} C & D & D \\ D & C & D \\ D & D & C \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

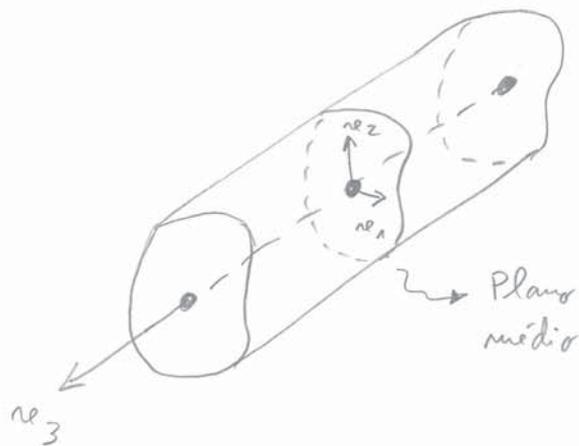
$$\left( C = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \right)$$

$$\left( D = \frac{E\nu}{(1+\nu)(1-2\nu)} \right)$$

$\rightarrow$ 

- Sólidos tridimensionais (Bricks)
- Estado Plano de Deformação (EPD)
- Estado Axissimétrico (EA)

# Estado Plano de Deformação (EPD)



- Corpo prismático.
- Dimensões segundo  $x_3$  muito superior às restantes ou deslocamentos segundo  $x_3$  impedidos.
- Acções paralelas ao plano médio
- Acções não variam com  $x_3$

Nestas circunstâncias, admite-se que os deslocamentos  $u$  apresentem as seguintes características:

$$\left. \begin{array}{l} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3 = 0 \end{array} \right\} \text{ não dependem de } x_3 \longrightarrow \text{ em todos os pontos do corpo}$$

$$\frac{\partial u_1}{\partial x_3} = 0 ; \frac{\partial u_3}{\partial x_1} = 0 \Rightarrow \gamma_{13} = \gamma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0$$

$$\frac{\partial u_2}{\partial x_3} = 0 ; \frac{\partial u_3}{\partial x_2} = 0 \Rightarrow \gamma_{23} = \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0$$

$$\frac{\partial u_3}{\partial x_3} = 0 \Rightarrow \epsilon_3 = \frac{\partial u_3}{\partial x_3} = 0$$

$$\underline{\sigma} = \underline{D} \underline{\epsilon}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \underline{D} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C & D & 0 \\ D & C & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$C = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$D = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Uma vez que,  $\epsilon_3 = -\frac{\nu}{E} \sigma_1 - \frac{\nu}{E} \sigma_2 + \frac{1}{E} \sigma_3 = 0$

$$\sigma_3 = \nu(\sigma_1 + \sigma_2)$$

Do caso geral é possível extrair a seguinte expressão:

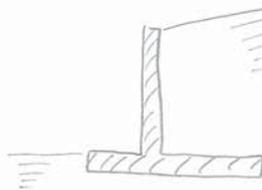
$$\sigma_3 = \frac{E\nu}{(1+\nu)(1-2\nu)} (\epsilon_1 + \epsilon_2)$$

Exemplos de estados planos de deformação (EPD):

Barragem gravidade



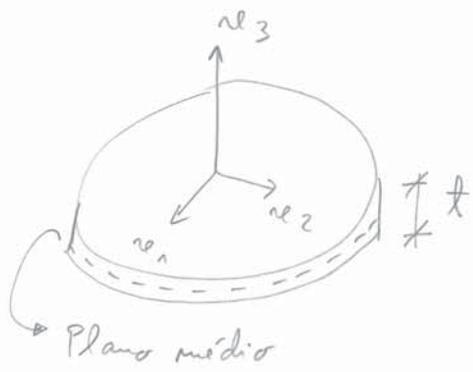
Muro de suporte



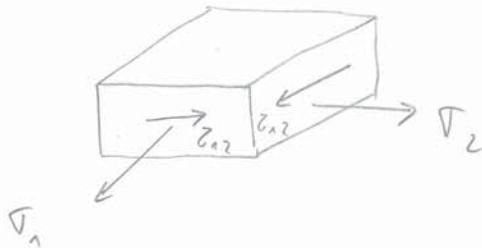
# Estado Plano de Tensão (EPT)

MEF-57

- Acções paralelas ao plano médio



espessura pequena em comparação com as restantes dimensões.



$$\sigma_3 = \tau_{23} = \tau_{31} = 0$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} A & B & B \\ B & A & B \\ B & B & A \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \cancel{\sigma_3} \end{bmatrix}$$

$$\left( A = \frac{1}{E} \right)$$

$$\left( B = -\frac{\nu}{E} \right)$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$$

$$\epsilon_3 = -\frac{\nu}{E} (\sigma_1 + \sigma_2)$$

Invertendo:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} C & D \\ D & C \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

$$\left( C = \frac{E}{1-\nu^2} \right)$$

$$\left( D = \frac{E\nu}{1-\nu^2} \right)$$

$$\underline{\sigma} = \underline{D} \underline{\epsilon}$$

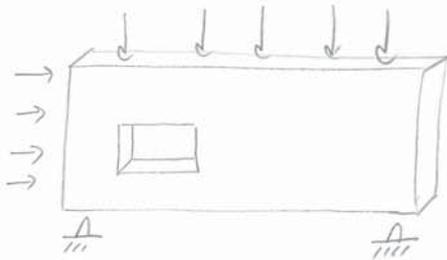
$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C & D & 0 \\ D & C & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\left( C = \frac{E}{1-\nu^2} \right)$$

$$\left( D = \frac{E\nu}{1-\nu^2} \right)$$

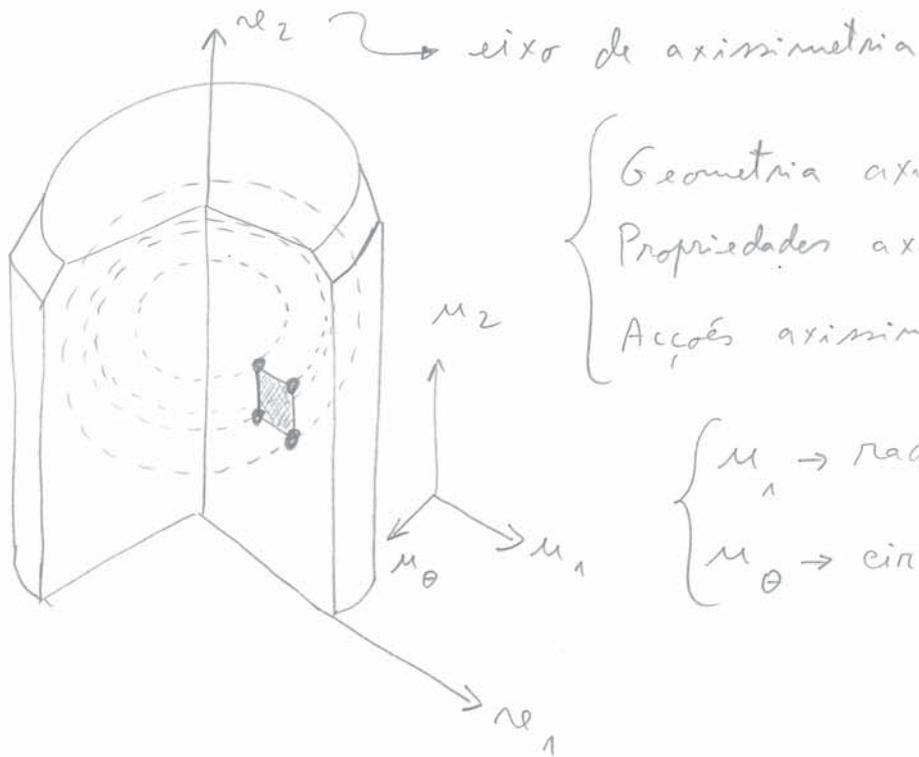
Exemplos de estados planos de tensão (EPT):

Viga parede



Console curta



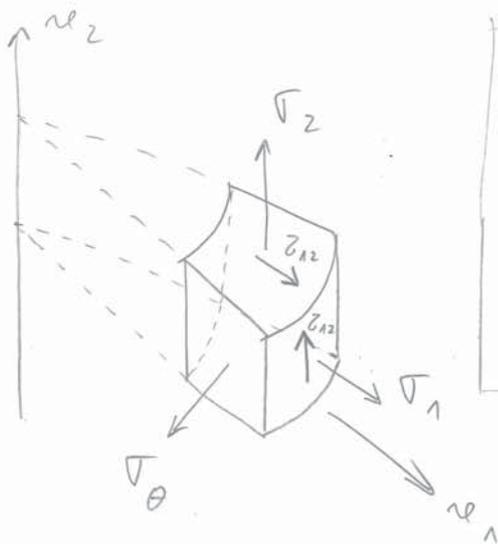


{ Geometria axissimétrica  
 Propriedades axissimétricas  
 Acções axissimétricas

$\begin{cases} u_1 \rightarrow \text{radial} \\ u_\theta \rightarrow \text{circunferencial} \end{cases}$

Considera-se uma malha de elementos finitos em  $(r_1, r_2)$

Elemento de volume  $dV$



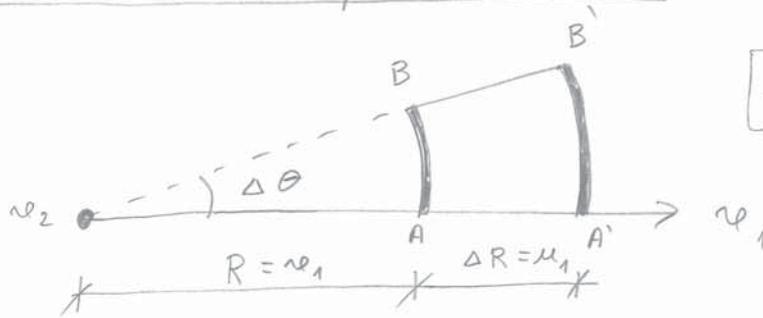
Devido à axissimetria:

$$\sigma_{1\theta} = \sigma_{2\theta} = 0$$

$$\gamma_{1\theta} = \gamma_{2\theta} = 0$$

# Deformação circunferencial $\epsilon_\theta$

MEF-60



$\mu_1 \rightarrow$  deslocamento radial

$$\epsilon_\theta = \frac{(A'B') - (AB)}{(AB)} = \frac{(R + \Delta R)\Delta\theta - R\Delta\theta}{R\Delta\theta} = \frac{\Delta R}{R}$$

$$\epsilon_\theta = \frac{\mu_1}{r_1}$$

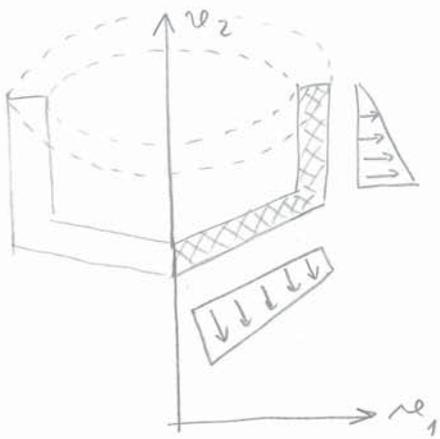
$$\begin{matrix} \epsilon \\ \sim \\ (4 \times 1) \end{matrix} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_\theta \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mu_1}{\partial r_1} \\ \frac{\partial \mu_2}{\partial r_2} \\ \frac{\mu_1}{r_1} \\ \frac{\partial \mu_1}{\partial r_2} + \frac{\partial \mu_2}{\partial r_1} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r_1} & 0 \\ 0 & \frac{\partial}{\partial r_2} \\ \frac{1}{r_1} & 0 \\ \frac{\partial}{\partial r_2} & \frac{\partial}{\partial r_1} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \underset{\sim}{L} \underset{\sim}{\mu} \quad \begin{matrix} (4 \times 2) & (2 \times 1) \end{matrix}$$

É utilizada a matriz  $\underset{\sim}{D}$  dos sólidos tridimensionais.

## Matriz de rigidez

$$\underset{\sim}{K} = \int_V \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} dV = \int_S \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \underbrace{(2\pi r_1)}_{\text{perímetro da circunferência de raio } r_1} dr_1 dr_2$$

perímetro  
da circunferência  
de raio  $r_1$



Depósito de água

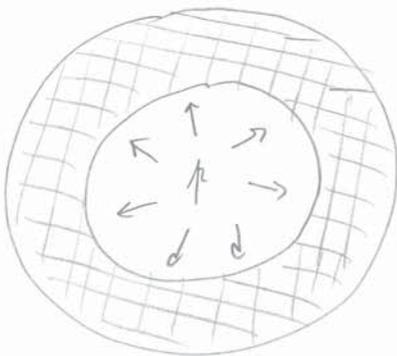


Pilar de secção circular variável

PROBLEMA

(13)

Considere o seguinte tubo de parede espessa:



$$\left[ \begin{array}{l} R_i = 2.5 \text{ m} \\ R_e = 3.8 \text{ m} \\ p = 2 \text{ KN/m}^2 \\ E = 200\,000 \text{ MPa} \\ \nu = 0.3 \end{array} \right.$$

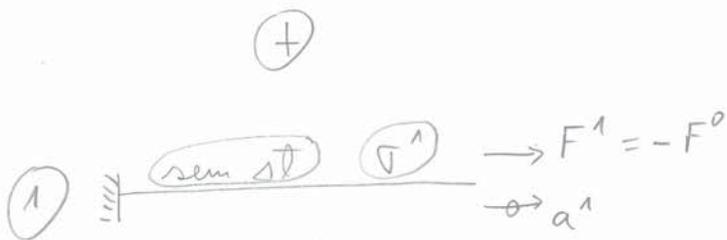
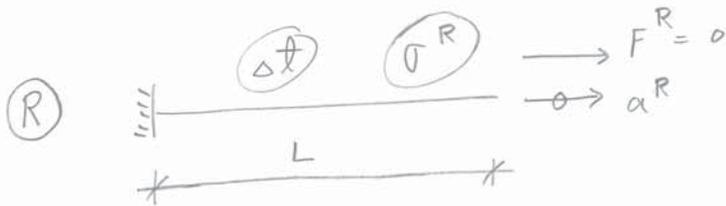
Calcule os esforços e os deslocamentos recorrendo ao método dos elementos finitos, formulando o problema

- como um estado plano de deformação
- como um estado axissimétrico

Fim da 14.ª aula

## Variações de temperatura

Exemplo uniaxial:



$K$  → matriz de rigidez

$\Delta t$  → variação de temperatura  
( $\oplus \Rightarrow$  aumento)

$\alpha$  → coeficiente de dilatação térmica ( $^{\circ}C^{-1}$ )

$E$  → módulo de elasticidade

$A$  → área da seção transversal

$\sigma$  → tensão

$\epsilon$  → deformação

$a$  → deslocamento nodal

$F$  → força nodal

$a^R, F^R$  → na estrutura real

$a^0, F^0$  → na estrutura bloqueada

$a^1, F^1$  → compensação calculada pelo método dos deslocamentos

(R) = (0) + (1)

$$a^R = a^0 + a^1 = a^1$$

$$F^R = F^0 + F^1 = 0$$

$$\sigma^R = \sigma^0 + \sigma^1$$

$$\epsilon^0 = -\alpha \Delta t$$

$$\sigma^0 = E \epsilon^0 \Rightarrow \sigma^0 = -E \alpha \Delta t$$

$$F^0 = A \sigma^0 \Rightarrow F^0 = -AE \alpha \Delta t$$

$$K a^1 = F^1 \Rightarrow \frac{EA}{L} a^1 = F^1 \Rightarrow \frac{EA}{L} a^1 = -F^0$$

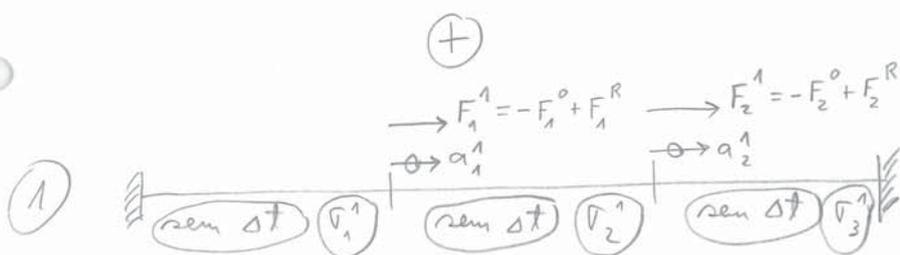
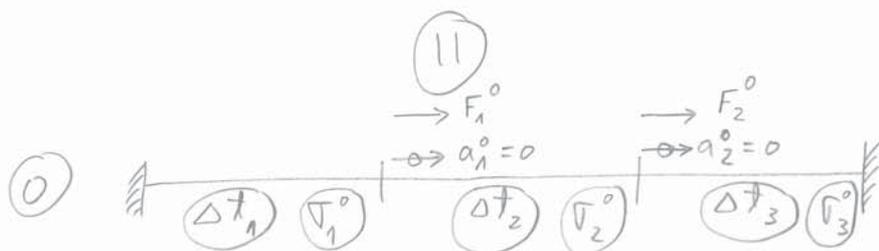
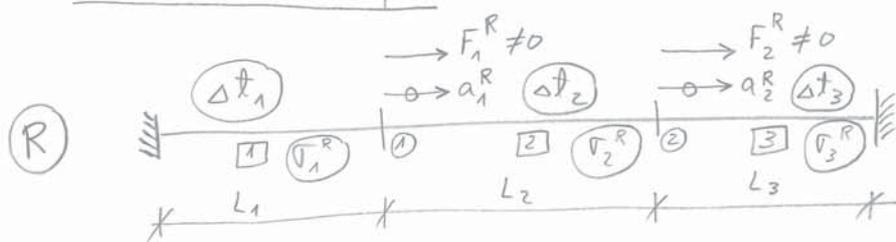
$$a^1 = \frac{L}{EA} AE \alpha \Delta t \Rightarrow a^1 = \alpha \Delta t L$$

$$\sigma^1 = \frac{E}{L} a^1 = E \alpha \Delta t$$

$$\sigma^R = \sigma^0 + \sigma^1 = 0$$

↳ neste exemplo simples

Outro exemplo:



$E, A, \alpha \rightarrow \text{constantes}$

$$\textcircled{R} = \textcircled{0} + \textcircled{1}$$

$$a_i^R = a_i^0 + a_i^1 = a_i^1 \quad (i=1, 2, 3)$$

$$F_i^R = F_i^0 + F_i^1 \quad (i=1, 2)$$

$$\sigma_j^R = \sigma_j^0 + \sigma_j^1 \quad (j=1, 2, 3)$$

$$\epsilon_j^0 = -\alpha \Delta t_j \quad (j=1, 2, 3)$$

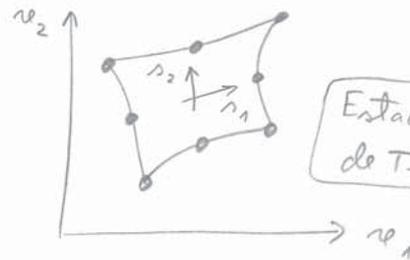
$$\sigma_j^0 = E \epsilon_j^0 \Rightarrow \sigma_j^0 = -E \alpha \Delta t_j \quad (j=1, 2, 3)$$

$$\begin{cases} F_1^0 = A \sigma_1^0 - A \sigma_2^0 \\ F_2^0 = A \sigma_2^0 - A \sigma_3^0 \end{cases}$$

$$\underset{\sim}{K} \underset{\sim}{a}^1 = \underset{\sim}{F}^1 \Rightarrow \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} a_1^1 \\ a_2^1 \end{bmatrix} = \begin{bmatrix} -F_1^0 + F_1^R \\ -F_2^0 + F_2^R \end{bmatrix} \rightsquigarrow \begin{cases} a_1^1 = \dots \\ a_2^1 = \dots \end{cases}$$

$$\begin{cases} \sigma_1^1 = \frac{E}{L_1} (a_1^1) \\ \sigma_2^1 = \frac{E}{L_2} (-a_1^1 + a_2^1) \\ \sigma_3^1 = \frac{E}{L_3} (-a_2^1) \end{cases} \rightarrow \sigma_j^R = \sigma_j^0 + \sigma_j^1 \quad (j=1, 2, 3)$$

# Elementos finitos



Estado Plano de Tensões

trabalho externo = trabalho interno

$$\tilde{a}^T \tilde{F} = \int_V \tilde{\epsilon}^T \tilde{\sigma} dV$$

$$\tilde{a}^T \tilde{F} = \int_V \tilde{a}^T \tilde{B}^T \tilde{\sigma} dV$$

$$\tilde{\epsilon} = \tilde{B} \tilde{a} \Rightarrow \tilde{\epsilon}^T = \tilde{a}^T \tilde{B}^T$$

$$\tilde{F} = \int_V \tilde{B}^T \tilde{\sigma} dV$$

→ cálculo das forças nodais equivalentes a um campo de tensões  $\tilde{\sigma}$

$$\tilde{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} \rightarrow \tilde{\epsilon}^0 = \begin{bmatrix} -\alpha \Delta T \\ -\alpha \Delta T \\ 0 \end{bmatrix}$$

→ uma expansão uniforme não provoca distorções

Em cada elemento finito:

$$\tilde{\sigma}^0 = \tilde{D} \tilde{\epsilon}^0$$

$\tilde{D}$  → matriz de elasticidade  
 $t$  → espessura do elemento finito

$$\tilde{F}^0 = \int_V \tilde{B}^T \tilde{\sigma}^0 dV = \int_S \tilde{B}^T (\tilde{D} \tilde{\epsilon}^0) t ds$$

$$\tilde{F}^0 = \int_S \tilde{B}^T \tilde{D} \tilde{\epsilon}^0 t dv_1 dv_2 = \int_{-1}^{+1} \int_{-1}^{+1} \tilde{B}^T \tilde{D} \tilde{\epsilon}^0 t |\tilde{J}| ds_1 ds_2$$

recorrer à quadratura de Gauss

→ assembler no vetor solicitações global

$$\tilde{F}^1 = \tilde{F}^R - \tilde{F}^0$$

$$\tilde{K} \tilde{a}^1 = \tilde{F}^1 \Rightarrow \tilde{K} \tilde{a}^1 = \tilde{F}^R - \tilde{F}^0 \rightarrow \text{envolve todos os gdl da estrutura}$$

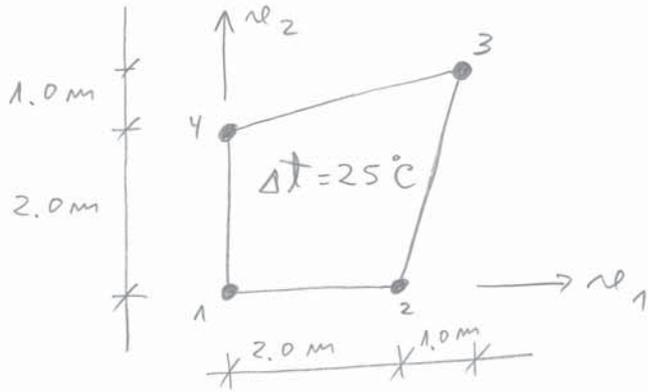
Depois de obtidos os deslocamentos globais  $\tilde{a}^1$ :

$$\left. \begin{aligned} \tilde{\sigma}^1 &= \tilde{D} \tilde{B} \tilde{a}^1 \\ \tilde{\sigma}^R &= \tilde{\sigma}^0 + \tilde{\sigma}^1 \end{aligned} \right\} \text{ao nível do elemento finito}$$

PROBLEMA (14)

MEF-65

Calcular as forças nodais equivalentes à seguinte ação (ver o problema 6):



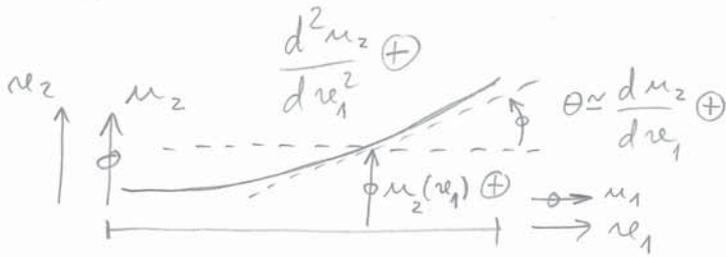
$$\left[ \begin{array}{l} t = 0.2 \text{ m (constante)} \\ \alpha = 10^{-5} \text{ } ^\circ\text{C}^{-1} \\ E = 200\,000 \text{ MPa} \\ \nu = 0.3 \end{array} \right.$$

Confirmar os resultados com o FEMIX.

Fim da 1.ª aula

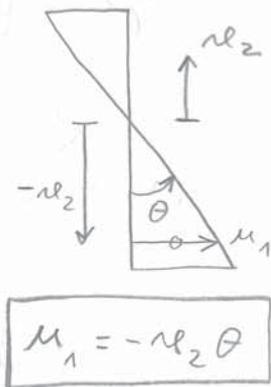
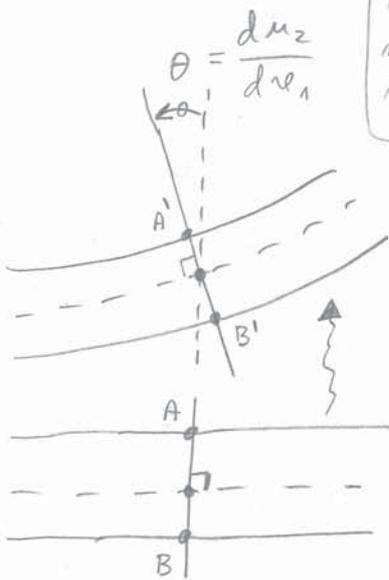
Vigas - formulação de Euler - Bernoulli

→ a deformação por esforços transversos não é considerada



$u_i \rightarrow$  deslocamento  
 $\frac{du_2}{dx_1} \rightarrow$  rotação ( $\theta$ )  
 $\frac{d^2u_2}{dx_1^2} \rightarrow$  curvatura

Após a deformação admite-se que as secções se mantêm planas e normais ao eixo da viga.



$I = \int_S \nu_2^2 ds \rightarrow$  momento de inércia da secção transversal da viga.  
 $\sigma_1 \rightarrow$  tensão normal  
 $\epsilon_1 \rightarrow$  extensão  
 $E \rightarrow$  módulo de elasticidade  
 $M \rightarrow$  momento fletor

$$\epsilon_1 = \frac{du_1}{dx_1} = \frac{d}{dx_1} (-\nu_2 \theta) = -\nu_2 \frac{d\theta}{dx_1} = -\nu_2 \frac{d^2u_2}{dx_1^2}$$

$$\sigma_1 = E \epsilon_1 = -E \nu_2 \frac{d\theta}{dx_1} = -E \nu_2 \frac{d^2u_2}{dx_1^2}$$

$$M = \int_S \sigma_1 \nu_2 ds = \int_S -E \nu_2^2 \frac{d^2u_2}{dx_1^2} ds$$

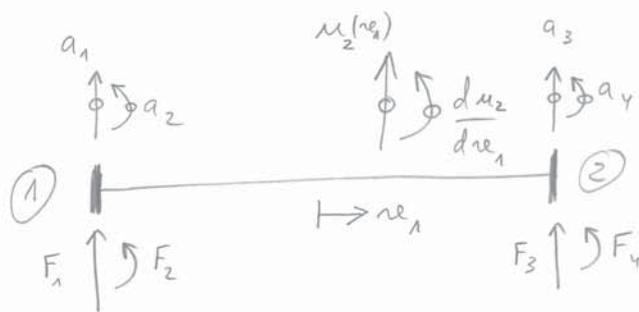
$\int_S$  secção transversal

$$M = -E \frac{d^2u_2}{dx_1^2} \int_S \nu_2^2 ds$$

$$M = -EI \frac{d^2u_2}{dx_1^2}$$

$\rightarrow$  momento  $\oplus \Rightarrow$  curvatura  $\ominus$

# Elemento finito de viga com dois nós



$$\textcircled{1} \rightarrow x_1 = -\frac{L}{2}$$

$$\textcircled{2} \rightarrow x_1 = \frac{L}{2}$$

$$\tilde{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \begin{array}{l} \rightarrow \text{deslocamento } (u_2) \text{ no ponto } \textcircled{1} \\ \rightarrow \text{rotação } \left(\frac{du_2}{dx_1}\right) \text{ no ponto } \textcircled{1} \\ \rightarrow \text{deslocamento } (u_2) \text{ no ponto } \textcircled{2} \\ \rightarrow \text{rotação } \left(\frac{du_2}{dx_1}\right) \text{ no ponto } \textcircled{2} \end{array}$$

$$\tilde{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{array}{l} \rightarrow \text{força no ponto } \textcircled{1} \\ \rightarrow \text{momento no ponto } \textcircled{1} \\ \rightarrow \text{força no ponto } \textcircled{2} \\ \rightarrow \text{momento no ponto } \textcircled{2} \end{array}$$

\* trabalho interno = trabalho externo  $\Leftrightarrow W_i = W_e$

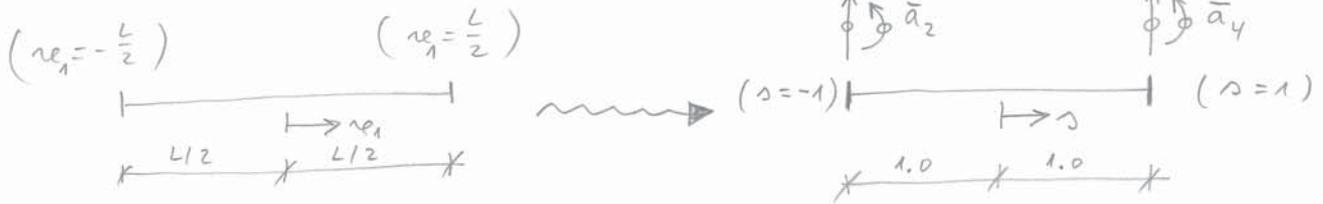
$$W_i = \int_V \sigma_1 \epsilon_1 dV = \int_L \int_S E x_2^2 \left(\frac{d^2 u_2}{dx_1^2}\right)^2 ds dL$$

$$W_i = \int_L E \left(\frac{d^2 u_2}{dx_1^2}\right)^2 \left[ \int_S x_2^2 ds \right] dL = \int_L E \left(\frac{d^2 u_2}{dx_1^2}\right)^2 I dL$$

$$W_e = \tilde{a}^T \tilde{F}$$

$$W_i = W_e \Leftrightarrow \int_{-L/2}^{+L/2} \left(\frac{d^2 u_2}{dx_1^2}\right) (EI) \left(\frac{d^2 u_2}{dx_1^2}\right) dx_1 = \tilde{a}^T \tilde{F}$$

## Transformações de coordenadas



$$x_{e1} = \frac{L}{2} s$$

$$J = \frac{dx_{e1}}{ds} = \frac{L}{2} \rightarrow \text{neste exemplo o Jacobiano é constante}$$

$$\tilde{\mathbf{a}} = \begin{bmatrix} a_1 \\ \bar{a}_2 \\ a_3 \\ \bar{a}_4 \end{bmatrix} \rightarrow \begin{array}{l} u_2(s) \text{ para } s = -1 \rightarrow \text{coincide com } u_2(x_{e1}) \text{ para } x_{e1} = -\frac{L}{2} \\ \frac{du_2}{ds} \text{ para } s = -1 \rightarrow \text{é diferente de } \frac{du_2}{dx_{e1}} \text{ para } x_{e1} = -\frac{L}{2} \\ u_2(s) \text{ para } s = 1 \rightarrow \text{coincide com } u_2(x_{e1}) \text{ para } x_{e1} = \frac{L}{2} \\ \frac{du_2}{ds} \text{ para } s = 1 \rightarrow \text{é diferente de } \frac{du_2}{dx_{e1}} \text{ para } x_{e1} = \frac{L}{2} \end{array}$$

$$\frac{du_2}{ds} = \frac{du_2}{dx_{e1}} \frac{dx_{e1}}{ds} \Rightarrow \frac{du_2}{ds} = \frac{L}{2} \frac{du_2}{dx_{e1}}$$

$$\left\{ \begin{array}{l} \text{Para } s = -1 \rightarrow \bar{a}_2 = \frac{L}{2} a_2 \\ \text{Para } s = 1 \rightarrow \bar{a}_4 = \frac{L}{2} a_4 \end{array} \right.$$

$$\tilde{\mathbf{a}} = \begin{bmatrix} a_1 \\ \frac{L}{2} a_2 \\ a_3 \\ \frac{L}{2} a_4 \end{bmatrix}$$

# Interpolação Hermítica no referencial $\xi$

MEF-69

4 grandezas nodais  $(a_1, \bar{a}_2, a_3, \bar{a}_4) \Rightarrow$  polinômio com 4 termos

$$u_2(\xi) = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \alpha_4 \xi^3$$

$$u_2(\xi) = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \Rightarrow u_2 = \underset{(1 \times 1)}{P} \underset{(1 \times 4)}{\sim} \underset{(4 \times 1)}{\sim} \alpha$$

$$\frac{du_2}{d\xi} = \alpha_2 + 2\alpha_3 \xi + 3\alpha_4 \xi^2$$

$$\xi = -1 \Rightarrow \begin{cases} u_2 = a_1 \Rightarrow \alpha_1 + \alpha_2(-1) + \alpha_3(-1)^2 + \alpha_4(-1)^3 = a_1 \\ \frac{du_2}{d\xi} = \bar{a}_2 \Rightarrow \alpha_2 + 2\alpha_3(-1) + 3\alpha_4(-1)^2 = \bar{a}_2 \end{cases}$$

$$\xi = +1 \Rightarrow \begin{cases} u_2 = a_3 \Rightarrow \alpha_1 + \alpha_2(1) + \alpha_3(1)^2 + \alpha_4(1)^3 = a_3 \\ \frac{du_2}{d\xi} = \bar{a}_4 \Rightarrow \alpha_2 + 2\alpha_3(1) + 3\alpha_4(1)^2 = \bar{a}_4 \end{cases}$$

$$\begin{cases} a_1 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 \\ \bar{a}_2 = \alpha_2 - 2\alpha_3 + 3\alpha_4 \\ a_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ \bar{a}_4 = \alpha_2 + 2\alpha_3 + 3\alpha_4 \end{cases}$$

$$\begin{bmatrix} a_1 \\ \bar{a}_2 \\ a_3 \\ \bar{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \Rightarrow \underset{\sim}{\bar{a}} = \underset{\sim}{C} \underset{\sim}{\alpha}$$

Invertendo a matriz  $C$  obtem-se:

MEF-70

$$C^{-1} = \begin{bmatrix} 1/2 & 1/4 & 1/2 & -1/4 \\ -3/4 & -1/4 & 3/4 & -1/4 \\ 0 & -1/4 & 0 & 1/4 \\ 1/4 & 1/4 & -1/4 & 1/4 \end{bmatrix}$$

$$x = C^{-1} \bar{a}$$

$$u_2 = P x \Rightarrow u_2 = P C^{-1} \bar{a}$$

$$u_2 = \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/4 & 1/2 & -1/4 \\ -3/4 & -1/4 & 3/4 & -1/4 \\ 0 & -1/4 & 0 & 1/4 \\ 1/4 & 1/4 & -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} a_1 \\ \frac{L}{2} a_2 \\ a_3 \\ \frac{L}{2} a_4 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} \frac{1}{2} - \frac{3}{4}\rho + \frac{1}{4}\rho^3 & \frac{L}{2} \left( \frac{1}{4} - \frac{1}{4}\rho - \frac{1}{4}\rho^2 + \frac{1}{4}\rho^3 \right) & \frac{1}{2} + \frac{3}{4}\rho - \frac{1}{4}\rho^3 & \frac{L}{2} \left( -\frac{1}{4} - \frac{1}{4}\rho + \frac{1}{4}\rho^2 + \frac{1}{4}\rho^3 \right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \Rightarrow u_2 = N a$$

(1x1) (1x4) (4x1)

$$\frac{du_2}{ds} = \frac{dN}{ds} \tilde{a} = \begin{bmatrix} -\frac{3}{4} + \frac{3}{4}s^2 & \frac{L}{2} \left(-\frac{1}{4} - \frac{1}{2}s + \frac{3}{4}s^2\right) & \frac{3}{4} - \frac{3}{4}s^2 & \frac{L}{2} \left(-\frac{1}{4} + \frac{1}{2}s + \frac{3}{4}s^2\right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\frac{d^2u_2}{ds^2} = \frac{d^2N}{ds^2} \tilde{a} = \begin{bmatrix} \frac{3}{2}s & \frac{L}{2} \left(-\frac{1}{2} + \frac{3}{2}s\right) & -\frac{3}{2}s & \frac{L}{2} \left(\frac{1}{2} + \frac{3}{2}s\right) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$\frac{d^2N_1}{ds^2}$                        $\frac{d^2N_2}{ds^2}$                        $\frac{d^2N_3}{ds^2}$                        $\frac{d^2N_4}{ds^2}$

Considerando apenas a função de forma associada ao nó  $j$  ( $N_j$ ):

$$\frac{dN_j}{ds} = \frac{dN_j}{dre_1} \frac{dre_1}{ds}$$

$$\frac{d^2N_j}{ds^2} = \frac{d}{dre_1} \left( \frac{dN_j}{dre_1} \frac{dre_1}{ds} \right) \frac{dre_1}{ds} = \frac{d^2N_j}{dre_1^2} \left( \frac{dre_1}{ds} \right)^2 = \frac{d^2N_j}{dre_1^2} \left( \frac{L}{2} \right)^2$$

$$\frac{d^2N_j}{dre_1^2} = \frac{4}{L^2} \frac{d^2N_j}{ds^2}$$

$$u_2 = N a$$

$$\frac{d^2 u_2}{d x_1^2} = \frac{d^2 N}{d x_1^2} a = \underbrace{\left[ \frac{d^2 N_1}{d x_1^2} \mid \frac{d^2 N_2}{d x_1^2} \mid \frac{d^2 N_3}{d x_1^2} \mid \frac{d^2 N_4}{d x_1^2} \right]}_B \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\boxed{\frac{d^2 u_2}{d x_1^2} = B a}$$

$$B = \left[ \frac{4}{L^2} \frac{d^2 N_1}{d s^2} \mid \frac{4}{L^2} \frac{d^2 N_2}{d s^2} \mid \frac{4}{L^2} \frac{d^2 N_3}{d s^2} \mid \frac{4}{L^2} \frac{d^2 N_4}{d s^2} \right]$$

$$B = \left[ \frac{6}{L^2} \curvearrowright \mid -\frac{1}{L} + \frac{3}{L} \curvearrowright \mid -\frac{6}{L^2} \curvearrowright \mid \frac{1}{L} + \frac{3}{L} \curvearrowright \right]$$

$\checkmark_{e1}$  MEF-67 :

$$W_i = W_e \Leftrightarrow \int_{-L/2}^{+L/2} \left( \frac{d^2 u_2}{d x_1^2} \right)^T (EI) \left( \frac{d^2 u_2}{d x_1^2} \right) d x_1 = a^T F$$

$$\int_{-L/2}^{+L/2} a^T B^T (EI) B a d x_1 = a^T F$$

$$\left[ \int_{-L/2}^{+L/2} B^T (EI) B d x_1 \right] a = F$$

$K \rightarrow$  matriz de rigidez do elemento de viga

## Mudança de variável

MEF-73

$$\underset{\sim}{K} = \int_{-L/2}^{+L/2} \underset{\sim}{B}^T (EI) \underset{\sim}{B} \, d\eta_1 = \int_{-1}^{+1} \underset{\sim}{B}^T (EI) \underset{\sim}{B} \left( \frac{d\eta_1}{ds} \right) ds$$

(4x4)      (4x4) (1x1) (1x4)

$$\underset{\sim}{K} = \int_{-1}^{+1} \begin{bmatrix} \frac{6}{L^2} \eta \\ -\frac{1}{L} + \frac{3}{L} \eta \\ -\frac{6}{L^2} \eta \\ \frac{1}{L} + \frac{3}{L} \eta \end{bmatrix} (EI) \begin{bmatrix} \frac{6}{L^2} \eta & -\frac{1}{L} + \frac{3}{L} \eta & -\frac{6}{L^2} \eta & \frac{1}{L} + \frac{3}{L} \eta \end{bmatrix} \left( \frac{L}{2} \right) ds$$

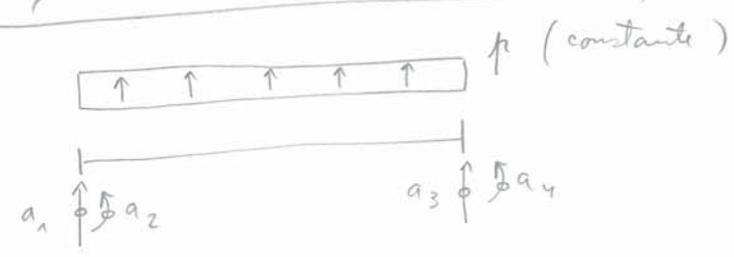
Exemplos:

$$K_{11} = \int_{-1}^{+1} \left( \frac{6}{L^2} \eta \right) (EI) \left( \frac{6}{L^2} \eta \right) \left( \frac{L}{2} \right) ds = EI \int_{-1}^{+1} \frac{18}{L^3} \eta^2 ds = \frac{12EI}{L^3}$$

$$K_{12} = \int_{-1}^{+1} \left( \frac{6}{L^2} \eta \right) (EI) \left( -\frac{1}{L} + \frac{3}{L} \eta \right) \left( \frac{L}{2} \right) ds = EI \int_{-1}^{+1} \left( -\frac{3}{L^2} \eta + \frac{9}{L^2} \eta^2 \right) ds = \frac{6EI}{L^2}$$

$$K_{22} = \int_{-1}^{+1} \left( -\frac{1}{L} + \frac{3}{L} \eta \right) (EI) \left( -\frac{1}{L} + \frac{3}{L} \eta \right) \left( \frac{L}{2} \right) ds = \frac{4EI}{L}$$

Ação distribuída na viga



$$\tilde{F} = \int_{-L}^{+L} \tilde{N}^T p \, ds = \int_{-L}^{+L} \tilde{N}^T p \frac{ds}{L} \, ds$$

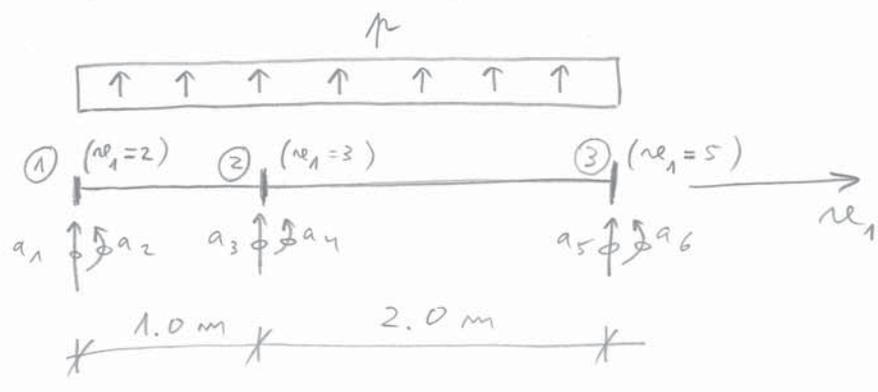
(4x1)                      (4x1)

$$F_1 = \int_{-L}^{+L} N_1 p \frac{L}{2} \, ds = \int_{-L}^{+L} \left( \frac{1}{2} - \frac{3}{4}s + \frac{1}{4}s^3 \right) p \frac{L}{2} \, ds = \frac{pL}{2}$$

$$F_2 = \int_{-L}^{+L} N_2 p \frac{L}{2} \, ds = \int_{-L}^{+L} \frac{L}{2} \left( \frac{1}{4} - \frac{1}{4}s - \frac{1}{4}s^2 + \frac{1}{4}s^3 \right) p \frac{L}{2} \, ds = \frac{pL^2}{12}$$

PROBLEMA (15) → Euler-Bernoulli:

Calcular a matriz de rigidez e o vetor solicitação correspondentes ao seguinte elemento finito (ver MEF-15):

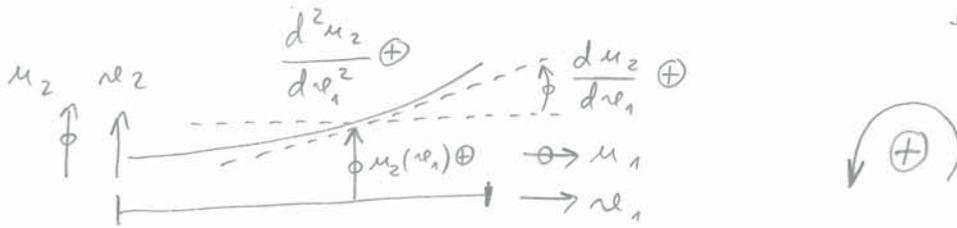


$$\left[ \begin{array}{l} E = 200\,000 \text{ MPa} \\ I = 10^{-6} \text{ m}^4 \\ p = 0.1 \text{ MN/m} \end{array} \right.$$

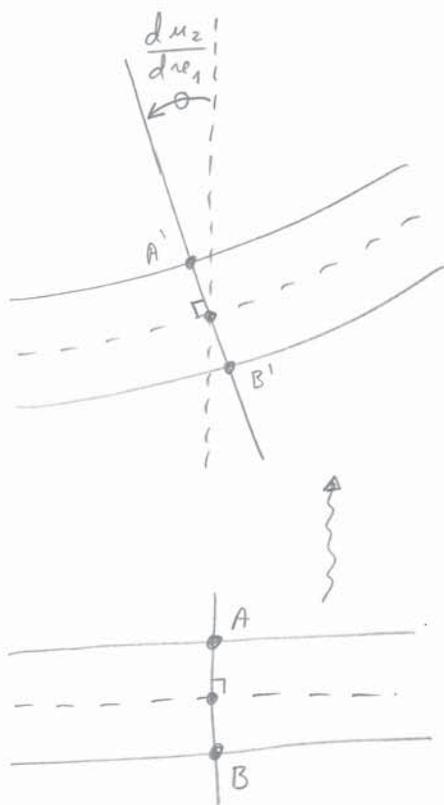
Fim da 16.ª aula

Vigas - formulações de Timoshenko

→ a deformação por esforço transverso é considerada

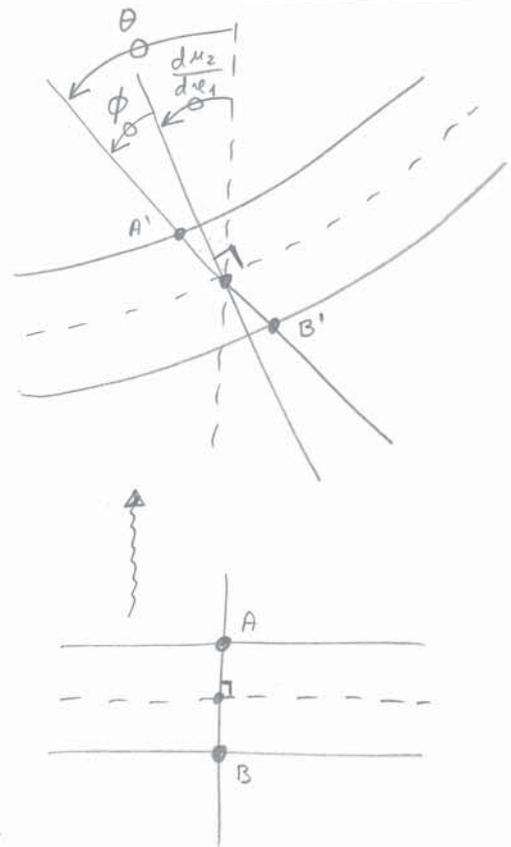


Euler - Bernoulli

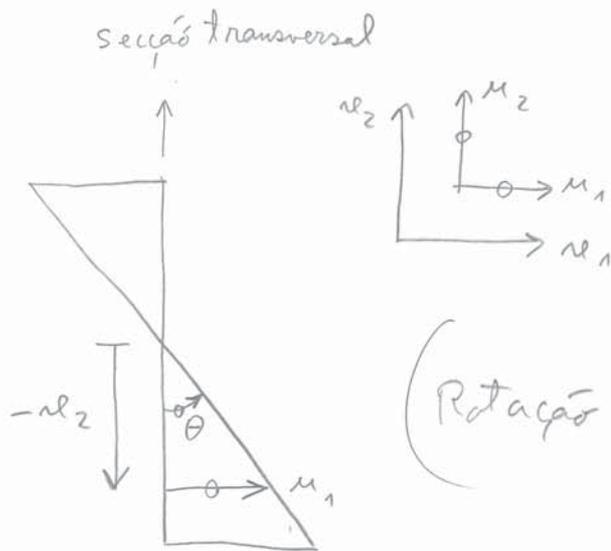


$$\theta = \frac{du_2}{dx_1}$$

Timoshenko



$$\theta = \frac{du_2}{dx_1} + \phi$$



(Rotação  $\oplus \Rightarrow M_1 \oplus$  em  $r_2 \ominus$ )

$$u_1 = -r_2 \theta$$

$$\epsilon_1 = \frac{du_1}{dr_1} = \frac{d}{dr_1} (-r_2 \theta) \Rightarrow \epsilon_1 = -r_2 \frac{d\theta}{dr_1}$$

$$\gamma_{12} = \frac{du_1}{dr_2} + \frac{du_2}{dr_1} = \frac{d}{dr_2} (-r_2 \theta) + \frac{du_2}{dr_1} = -\theta + \frac{du_2}{dr_1}$$

Uma vez que:  $-\theta + \frac{du_2}{dr_1} = -\phi$  (ver MEF-75)

$$\gamma_{12} = -\phi$$

trabalho interno = trabalho externo

MEF-77

$$W_i = W_e \Rightarrow W_i^F + W_i^S = W_e$$

trabalho interno associado à flexão

trabalho interno associado ao corte

$$W_i^F = \int_V \tau_{11} \epsilon_{11} dV = \int_V E \epsilon_{11}^2 dV =$$

$$\tau_{11} = E \epsilon_{11}$$

$$= \int_L \int_S E r_{12}^2 \left( \frac{d\theta}{dx_1} \right)^2 ds dL =$$

$$= \int_L E \left( \frac{d\theta}{dx_1} \right)^2 \left[ \int_S r_{12}^2 ds \right] dL = \int_L EI \left( \frac{d\theta}{dx_1} \right)^2 dL$$

$$W_i^F = \int_L \left( \frac{d\theta}{dx_1} \right) (EI) \left( \frac{d\theta}{dx_1} \right) dL$$

$$W_i^S = \int_V \tau_{12} \delta_{12} dV = \int_V G \delta_{12}^2 dV =$$

$$= \int_V G (-\phi)^2 dV = \int_L \int_S G \phi^2 ds dL =$$

$$= \int_L G \phi^2 \left[ \int_S ds \right] dL = \int_L GA' \phi^2 dL$$

$$W_i^S = \int_L \phi (GA') \phi dL$$

$$\tau_{12} = G \delta_{12}$$

→ módulo de distorção

$$\delta_{12} = -\phi$$

(ver MEF-76)

$A'$  → área efetiva de corte

## Momento fletor (M)

MEF-78

$$M = \int_S \sigma_{11} x_2 ds = \int_S E \epsilon_1 x_2 ds =$$

$$= \int_S E \left( -x_2 \frac{d\theta}{dx_1} \right) x_2 ds = - \int_S E \frac{d\theta}{dx_1} x_2^2 ds =$$

$$= - \frac{d\theta}{dx_1} E \int_S x_2^2 ds \Rightarrow \boxed{M = -EI \frac{d\theta}{dx_1}}$$

$\left( \frac{M \oplus}{\text{momento } \oplus \Rightarrow \text{curvatura } \ominus} \right)$

## Esforço transversal (Q)

$$Q = \int_S \tau_{12} ds = \int_S G \gamma_{12} ds = \int_S G (-\phi) ds =$$

$$= -G \phi \int_S ds \Rightarrow \boxed{Q = -GA' \phi}$$

## Elemento finito com 2 nós

$$a_1 \uparrow \downarrow a_2 \quad a_3 \uparrow \downarrow a_4$$

→ Os campos de deslocamentos e rotações são interpolados separadamente

$$\begin{cases} u_2 = N_1 a_1 + N_2 a_3 \Rightarrow \frac{du_2}{dx_1} = \frac{dN_1}{dx_1} a_1 + \frac{dN_2}{dx_1} a_3 \\ \theta = N_1 a_2 + N_2 a_4 \Rightarrow \frac{d\theta}{dx_1} = \frac{dN_1}{dx_1} a_2 + \frac{dN_2}{dx_1} a_4 \end{cases}$$

$$\phi = \theta - \frac{du_2}{dx_1} \Rightarrow \phi = N_1 a_2 + N_2 a_4 - \frac{dN_1}{dx_1} a_1 - \frac{dN_2}{dx_1} a_3$$

→ (ver MEF-75)

$$W_i = W_i^F + W_i^S = \int_L \left( \frac{d\theta}{dx_1} \right) (EI) \left( \frac{d\theta}{dx_1} \right) dx_1 + \int_L \phi (GA') \phi dx_1 = \boxed{\text{MEF-79}}$$

$$= \int_L \left( \frac{dN_1}{dx_1} a_2 + \frac{dN_2}{dx_1} a_4 \right) (EI) \left( \frac{dN_1}{dx_1} a_2 + \frac{dN_2}{dx_1} a_4 \right) dx_1 +$$

$$+ \int_L \left( -\frac{dN_1}{dx_1} a_1 + N_1 a_2 - \frac{dN_2}{dx_1} a_3 + N_2 a_4 \right) (GA') \left( -\frac{dN_1}{dx_1} a_1 + N_1 a_2 - \frac{dN_2}{dx_1} a_3 + N_2 a_4 \right) dx_1 =$$

$$\bullet = \int_L \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{dN_1}{dx_1} \\ 0 \\ \frac{dN_2}{dx_1} \end{bmatrix} [EI] \begin{bmatrix} 0 & \frac{dN_1}{dx_1} & 0 & \frac{dN_2}{dx_1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} dx_1 +$$

$$+ \int_L \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} -\frac{dN_1}{dx_1} \\ N_1 \\ -\frac{dN_2}{dx_1} \\ N_2 \end{bmatrix} [GA'] \begin{bmatrix} -\frac{dN_1}{dx_1} & N_1 & -\frac{dN_2}{dx_1} & N_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} dx_1 =$$

$$= \int_L \tilde{a}^T \tilde{B}_F^T \tilde{D}_F \tilde{B}_F \tilde{a} dx_1 + \int_L \tilde{a}^T \tilde{B}_S^T \tilde{D}_S \tilde{B}_S \tilde{a} dx_1$$

$$W_{\tilde{a}} = W_e \Rightarrow \tilde{a}^T \left[ \int_L \tilde{B}_F^T \tilde{D}_F \tilde{B}_F d\tilde{v}_1 + \int_L \tilde{B}_S^T \tilde{D}_S \tilde{B}_S d\tilde{v}_1 \right] \tilde{a} = \tilde{a}^T \tilde{F}$$

$$\tilde{K} \tilde{a} = \tilde{F}$$

$$\tilde{K} = \tilde{K}_F + \tilde{K}_S$$

$$\begin{cases} \tilde{K}_F = \int_L \tilde{B}_F^T \tilde{D}_F \tilde{B}_F d\tilde{v}_1 \\ \tilde{K}_S = \int_L \tilde{B}_S^T \tilde{D}_S \tilde{B}_S d\tilde{v}_1 \end{cases}$$

$$\tilde{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\tilde{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

$$\tilde{D}_F = [EI] \quad (1 \times 1)$$

$$\tilde{D}_S = [GA'] \quad (1 \times 1)$$

$$\tilde{B}_F = \begin{bmatrix} 0 & \frac{dN_1}{d\tilde{v}_1} & 0 & \frac{dN_2}{d\tilde{v}_1} \end{bmatrix}$$

$$\tilde{B}_S = \begin{bmatrix} -\frac{dN_1}{d\tilde{v}_1} & N_1 & -\frac{dN_2}{d\tilde{v}_1} & N_2 \end{bmatrix}$$

### Transformação de coordenadas

$$\begin{array}{c} x \quad L/2 \quad x \quad L/2 \quad x \\ \hline | \quad \quad \quad | \\ (v_1 = -L/2) \quad \rightarrow \quad v_1 \quad (v_1 = L/2) \end{array}$$



$$\begin{array}{c} x \quad 1 \quad x \quad 1 \quad x \\ \hline | \quad \quad \quad | \\ (s = -1) \quad \rightarrow \quad s \quad (s = 1) \end{array}$$

$$v_1 = \frac{L}{2} s \Rightarrow \frac{dv_1}{ds} = \frac{L}{2} \Rightarrow \frac{ds}{dv_1} = \frac{2}{L}$$

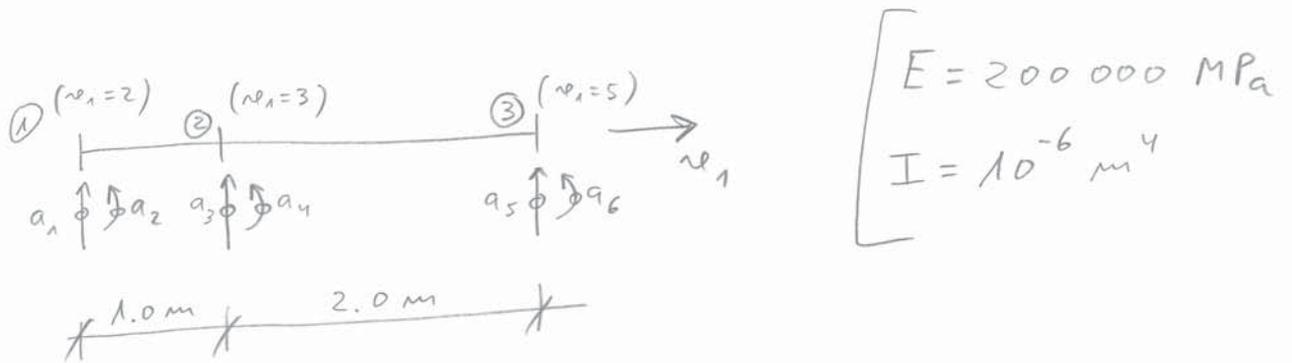
$$\frac{dN_i}{dv_1} = \frac{dN_i}{ds} \frac{ds}{dv_1} \Rightarrow \frac{dN_i}{dv_1} = \frac{2}{L} \frac{dN_i}{ds}$$

$$\int_{-L/2}^{+L/2} \boxed{\otimes} dv_1 = \int_{-1}^{+1} \boxed{\otimes} \frac{dv_1}{ds} ds = \int_{-1}^{+1} \boxed{\otimes} \frac{L}{2} ds$$

PROBLEMA 16 → Timoshenko

MEF-81

Calcular a matriz de rigidez correspondente ao seguinte elemento finito (ver MEF-15):



Fim da 17ª aula

FIM