SECOND-ORDER OPTIMIZATION OF FRAMES WITH NONLINEAR BEHAVIOR

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PROBLEM

• Minimize the cost of a portal frame

FRAME BEHAVIOR

• Liner elastic between plastic hinges

• Plastic hinges with limited plastic rotation

• Plastic hinges are automatically located in the most critical positions
OPTIMIZATION APPROACH

- Solve of a nonlinear program
- Second-order approximation
- Integrated formulation
- No sensitivity analysis
- All the problem variables are present in the nonlinear program
OPTIMIZATION SOFTWARE

- NEWTOP
- General purpose code
- Lagrange-Newton method
- Symbolic manipulation of all the functions
NONLINEAR PROGRAMMING

Minimize $f(x)$

subject to

$g(x) \leq 0 \quad \rightarrow \quad g_i(x) + s_i^2 = 0$

$h(x) = 0$

- Variables / functions \quad \rightarrow \quad \text{real and continuous}
- All the functions are generalized polynomials, such as:

$f(x) = 5.9x_1^2x_4^{-3} - 3.1x_2 + 2.7x_1^{-1}x_3x_5^2 - 1.8$
A symbolic manipulation is performed

Expression parsing and evaluation is simplified

Exact first and second derivatives can be easily calculated

All these operations can be efficiently performed
INPUT FILE

### Main title of the nonlinear program
Symmetric truss with two load cases (kN,cm)

Min.
\[ +565.685 \times t5^2 + 100 \times t8^2 \; ; \# \text{truss volume (cm3)} \]

s.t.i.c.
Min. area 4: \[- t4^2 + 0.15 < 0 \; ; \]

s.t.e.c.
Equil 16: \[ + 141.421 \times t5^2 \times \text{disp16} - 100 = 0 \; ; \]

END_OF_FILE

• All the software is coded in ANSI C
LAGRANGIAN

\[ L(X) = f(x) + \sum_{k=1}^{m} \lambda_{k}^{g} \left[ g_{k}(x) + s_{k}^{2} \right] + \sum_{k=1}^{p} \lambda_{k}^{h} h_{k}(x) \]

VARIABLES

\[ X = (s, \lambda_{s}, x, \lambda_{h}) \]

SOLUTION

• Stationary point of the Lagrangian
SYSTEM OF NONLINEAR EQUATIONS

\[
\begin{aligned}
\nabla L(X) = 0 \quad \Rightarrow \\
2s_i \lambda_i^g &= 0 \\
g_i + s_i^2 &= 0 \\
\frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \lambda_k^g \frac{\partial g_k}{\partial x_i} + \sum_{k=1}^{p} \lambda_k^h \frac{\partial h_k}{\partial x_i} &= 0 \\
h_i &= 0
\end{aligned}
\]

• The solution of the system is a KKT solution when

\[\lambda_i^g \geq 0\]
LAGRANGE-NEWTON METHOD

• The system of nonlinear equations

$$\nabla L(X) = 0$$

is solved by the Newton method

• In each iteration the following system of linear equations has to be solved

$$H(X^{q-1}) \Delta X^q + \nabla L(X^{q-1}) = 0$$
**HESSIAN MATRIX**

\[
H =
\begin{array}{cccc}
\text{Diag}(2\lambda^g_i) & \text{Diag}(2s_i) & 0 & 0 \\
0 & \frac{\partial g_i}{\partial x_j} & 0 & \\
0 & 0 & \frac{\partial h_j}{\partial x_i} & 0 \\
\text{SYMMETRIC} & & & 0 \\
\end{array}
\]

\[
\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=1}^m \lambda^g_k \frac{\partial^2 g_k}{\partial x_i \partial x_j} + \sum_{k=1}^p \lambda^h_k \frac{\partial^2 h_k}{\partial x_i \partial x_j}
\]
Hessian Matrix Sparsity Pattern

\[ H = \sim \]
SYSTEM OF LINEAR EQUATIONS

- Gaussian elimination
  - adapted to the sparsity pattern of the Hessian matrix

- Conjugate gradients
  - diagonal preconditioning
  - adapted to an indefinite Hessian matrix

LINE SEARCH

\[
X^q = X^{q-1} + \alpha \Delta X^q
\]
NEWTOP COMPUTER CODE

- All the variables are scaled
- Constraints are normalized
- Elementary equality constraints are substituted:
  \[ x_i = c x_j \quad \text{or} \quad x_i = c \]
- The NLP is simplified
- Large scale problems can be solved
PORTAL FRAME

- Cost minimization
- Independent design variables \(\Rightarrow\) cross section parameters
NONLINEAR MATERIAL BEHAVIOR

• Linear-perfectly plastic behavior

• Linear-constant moment-curvature diagram

\[ M_p = \sigma_{\text{max}} BH^2 / 4 \]

\[ M_p = \sigma_{\text{max}} w (3B^2 - 6Bw + 4w^2) / 2 \]
NONLINEAR MATERIAL BEHAVIOR

- Plastic deformations concentrated in plastic hinges
- Plastic hinge rotation may be limited
- Collapse mechanism may not be reached
- Linear behavior between plastic hinges
STRUCTURAL DISCRETIZATION

\[ K_a \, \Delta A = F_a + P_a \]

\[ K_b \, \Delta B = F_b + P_b \]

- Hinge C \( \Rightarrow \) point of max. bending moment
EQUILIBRIUM EQUATIONS

\[ \sum_{a} F_a + \sum_{b} F_b' + \sum_{b} F_b'' + \cdots = Q + R \]

- Reactions are only present in constrained dof’s
NON LINEAR PROGRAM

- Objective function: \( f(x) = \sum_{i=1}^{NB} c_i A_i L_i \)

- Equality constraints:
  - beam behavior \( K_a d_a = F_a + P_a \)
  - equilibrium \( F_a ' + \cdots + F_b ' ' + \cdots = Q + R \)
  - compatibility \( d_{a1} = D_{N1} \)
  - cross section properties \( A, I, M_p = \cdots \)
  - beam length \( L = a + b \)
• Equality constraints (cont.):

- plastic hinge rotation  \( \theta_A = d_{a3} - D_{N3} \)

- elastic-plastic complementary in each plastic hinge:
  \[
  \theta_A = 0 \quad \text{or} \quad F_{a3} = M_p \quad \text{or} \quad F_{a3} = -M_p \quad \Rightarrow \quad \theta_A \left( M_p^2 - F_{a3}^2 \right) = 0
  \]

- nodal displacement  \( D_j = \overline{D} \) (only at prescribed dof’s)

- reaction  \( R_k = 0 \) (only at non prescribed dof’s)

- null shear force in C  \( F_{a5} = 0 \) (locates C in the point of max. bending moment)
• Inequality constraints:

  ◆ side constraints \( x_{\text{min}} \leq x_i \leq x_{\text{max}} \)

  ◆ moments are limited by \( \pm M_p \) \( -M_p \leq F_{a3} \leq M_p \)

  ◆ limited plastic hinge rotation \( -\theta_{\text{min}} \leq \theta_c \leq \theta_{\text{max}} \)

  - limiting values depend on the type of material and on the shape of the cross section

  - crushing, brittle failure and local buckling can thus be avoided
NUMERICAL EXAMPLE

\[ E = 200 \text{ GPa} \]
\[ \sigma_{\text{max}} = 250 \text{ MPa} \]

\[ -0.01 \leq \theta \leq 0.01 \text{ rad} \]

\[ B_i \rightarrow \text{independent design variables} \]
NUMERICAL RESULTS

• Optimal solution - linear behavior
  ♦ Volume = 0.175 m³
  ♦ Horizontal displacement = 2.7 cm

• Optimal solution - nonlinear behavior
  ♦ Volume = 0.157 m³ (10 % smaller)
  ♦ Horizontal displacement = 5.8 cm (2 x)
CONCLUSIONS

- More realistic approach of the frame design problem
- Ultimate and serviceability conditions may be considered
- More economical structures can be designed
- Friendly user interface is still required
- Solving the nonlinear program is still a hard task