SECOND-ORDER SHAPE OPTIMIZATION
OF A STEEL BRIDGE

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PROBLEM

• Minimize the cost of a steel bridge
• Member sizing and shape optimization
STRUCTURAL BEHAVIOR

- Linear elastic
- Fixed nodes (normal direction)
- Local Euler buckling
- Portuguese structural codes
OPTIMIZATION APPROACH

- Nonlinear program
- Second-order approximation
- Integrated formulation
- All the problem variables are present in the nonlinear program
- No sensitivity analysis
OPTIMIZATION SOFTWARE

• NEWTOP
• General purpose code
• Lagrange-Newton method
• Symbolic manipulation of all the functions
NONLINEAR PROGRAMMING

Minimize $f(x)$

subject to

$g(x) \leq 0 \quad \Rightarrow \quad g_i(x) + s_i^2 = 0$

$h(x) = 0$

- Variables / functions $\rightarrow$ real and continuous
- All the functions are generalized polynomials, such as:

$$f(x) = 5.9 x_1^2 x_4^{-3} - 3.1 x_2 + 2.7 x_1^{-1} x_3 x_5^2 - 1.8$$
GENERALIZED POLYNOMIALS

\[ f(x) = 5.9 x_1^2 x_4^{-3} - 3.1 x_2 + 2.7 x_1^{-1} x_3 x_5^2 - 1.8 \]

- A symbolic manipulation is performed
- Expression parsing and evaluation is simplified
- Exact first and second derivatives can be easily calculated
- All these operations can be efficiently performed
INPUT FILE

### Main title of the nonlinear program

**Symmetric truss with two load cases (kN,cm)**

**Min.**

\[ +565.685 \times t_5^2 + 100 \times t_8^2 \; ; \; \# \; \text{truss volume (cm}^3) \]

**s.t.i.c.**

**Min. area 4:**

\[ - t_4^2 + 0.15 < 0 ; \]

**s.t.e.c.**

**Equil 16:**

\[ + 141.421 \times t_5^2 \times \text{disp}16 - 100 = 0 ; \]

**END_OF_FILE**

• All the software is coded in ANSI C
LAGRANGIAN

\[ L(X) = f(x) + \sum_{k=1}^{m} \lambda_k^g \left[ g_k(x) + s_k^2 \right] + \sum_{k=1}^{p} \lambda_k^h h_k(x) \]

VARIABLES

\[ X = (s, \lambda^g, x, \lambda^h) \]

SOLUTION

• Stationary point of the Lagrangian
SYSTEM OF NONLINEAR EQUATIONS

\[ \nabla L(X) = 0 \quad \Rightarrow \quad \begin{aligned}
2s_i \lambda_i^g &= 0 \\
g_i + s_i^2 &= 0 \\
\frac{\partial f}{\partial x_i} + \sum_{k=1}^{m} \lambda_k^g \frac{\partial g_k}{\partial x_i} + \sum_{k=1}^{p} \lambda_k^h \frac{\partial h_k}{\partial x_i} &= 0 \\
h_i &= 0
\end{aligned} \quad (i = 1, \ldots, m)

\quad (i = 1, \ldots, m)

\quad (i = 1, \ldots, n)

\quad (i = 1, \ldots, p)

• The solution of the system is a KKT solution when

\[ \lambda_i^g \geq 0 \]
LAGRANGE-NEWTON METHOD

• The system of nonlinear equations

\[ \nabla L (X) = 0 \]

is solved by the Newton method

• In each iteration the following system of linear equations has to be solved

\[ H \left( X^{q-1} \right) \Delta X^q + \nabla L \left( X^{q-1} \right) = 0 \]
HESSIAN MATRIX

\[
H = \begin{bmatrix}
\text{Diag}(2\lambda_i^g) & \text{Diag}(2s_i) & 0 & 0 \\
0 & \frac{\partial g_i}{\partial x_j} & 0 & 0 \\
0 & 0 & \frac{\partial h_j}{\partial x_i} & 0 \\
\text{SYM METRIC} & 0 & 0 & 0
\end{bmatrix}
\]

\[
\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=1}^{m} \lambda_k^g \frac{\partial^2 g_k}{\partial x_i \partial x_j} + \sum_{k=1}^{p} \lambda_k^h \frac{\partial^2 h_k}{\partial x_i \partial x_j}
\]
HESSIAN MATRIX SPARSITY PATTERN

\[ H = \sim \]
SYSTEM OF LINEAR EQUATIONS

- Gaussian elimination
  - adapted to the sparsity pattern of the Hessian matrix

- Conjugate gradients
  - diagonal preconditioning
  - adapted to an indefinite Hessian matrix
LINE SEARCH

\[ X^q = X^{q-1} + \alpha \Delta X^q \]

- The value of \( \alpha \) minimizes the error in \( \Delta X^q \) direction
  - the value of \( \alpha \) is often close to one
  - faster convergence
  - process may fail

- The value of \( \alpha \) is made considerably smaller (e.g. \( \alpha = 0.1 \))
  - stable convergence
  - more iterations - slower
NEWTOP COMPUTER CODE

- All the variables are scaled
- Constraints are normalized
- Elementary equality constraints are substituted:
  \[ x_i = c x_j \quad \text{or} \quad x_i = c \]
- The NLP is simplified
- Problems with a large number of variables can be solved
  (e.g., 4 000 design variables and 20 000 constraints)
TRUSS OPTIMIZATION

- Cost minimization (often similar to volume minimization)
- Sizing $\Rightarrow$ cross-sectional areas may change
- Shape optimization $\Rightarrow$ nodal coordinates may change

Simultaneously
VARIABLES

• Integrated formulation

• Design variables and behavior variables simultaneously present in the nonlinear program
  ♦ Cross-section dimensions (e.g., width, diameter, area)
  ♦ Some nodal coordinates
  ♦ Nodal displacements

\[ w = 0.63 \text{ cm} \]
SUBSTITUTED VARIABLES

• In most cases the area (A) and the moment of inertia (I) depend on a single parameter (B)

\[ A = C_0^A + C_1^A B + C_2^A B^2 \]
\[ I = C_0^I + C_1^I B + C_2^I B^2 + C_3^I B^3 + C_4^I B^4 \]

♦ Coefficients \( C_i^A \) and \( C_j^I \) are fixed

♦ Variables A and I can be substituted in all the functions that define
the mathematical program

\[ w = 0.63 \text{ cm} \]
ADDITIONAL VARIABLES

\[ k_{ij} = \cdots + EAL^{-1} + \cdots \]

\[ L = \sqrt{(x_{21} - x_{11})^2 + (x_{22} - x_{12})^2} \]

\[ -L^2 + x_{11}^2 + x_{12}^2 + x_{21}^2 + x_{22}^2 - 2x_{11}x_{21} - 2x_{12}x_{22} = 0 \]

♦ Additional variables \( \Rightarrow L_i \)

♦ Additional equality constraints \( \Rightarrow L_i \) definition
EQUILIBRIUM EQUATIONS

- Equality constraints:

\[ F_A + \cdots + F_B + \cdots = Q + R \]

- Reactions are only present in constrained dof’s
COMPATIBILITY EQUATIONS

- Variables \( d \) are substituted
- \( D_{Ni} \) is fixed in constrained dof’s
NON LINEAR PROGRAM

- Objective function: cost \[ f(x) = \sum_{i=1}^{NB} c_i A_i L_i \]

- Equality constraints:
  - for each bar with variable length:
    - one equation defining L
  - for each non-prescribed degree of freedom:
    - one equilibrium equation
• Inequality constraints:
  
  ♦ minimum width  \( B \geq B_{\text{min}} \)
  
  ♦ allowable stress (tension and compression)
  
  ♦ local Euler buckling
  
  ♦ side constraints in nodal coordinates  \( x_{\text{min}} \leq x_i \leq x_{\text{max}} \)
NUMERICAL EXAMPLE

Variables: \( A \), \( x \)

- \( \| \vec{F} \| = 200 \text{ kN} \)
- \( f_2 = 8 f_1 \)
- \( \sigma_{\text{max}} = 100 \text{ kN/cm}^2 \)

Svanberg’s solution confirmed
STEEL BRIDGE

Vertical distributed load

- Group I - horizontal bars
- Group II - diagonal bars

3.5 m

18.0 m

4.0 m

w = 0.63 cm

B
STEEL BRIDGE
OPTIMAL SHAPE

Initial shape

Optimal shape
NUMERICAL RESULTS

- Optimal solution - sizing only
  - Volume = 170 dm³

- Optimal solution - sizing and shape optimization
  - Volume = 146 dm³ (14 % smaller)
  - CPU time (PC): less than 10 seconds
CONCLUSIONS

• Significant economy in a structure that will be repeated
• Efficiency and accuracy
• Optimal structure is easy to build
• Friendly user interface is still required
• Move limits in nodal coordinates need some tuning